Analysis and Design of Microfinance Services: A Case of ROSCA

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Abstract

Rotating savings and credit association (ROSCA) is a well-known microfinance association widely used in many countries around the world with long histories. By considering extra profits that such a system can provide when compared to banking transactions, we develop optimization problems to achieve an optimal design of a ROSCA. We find that ROSCAs might attract investors when deposit and loan rates from formal banking systems are not favorable. Furthermore, optimal rates and optimal orders to maximize system outputs are reported.

KEYWORDS: microfinance; informal finance system; ROSCA; optimal design

1. Introduction

1.1 Motivation

Modern banking and financial systems have evolved very fast, however it is not so long ago that some sorts of informal financing operations, e.g. via friends and relatives, were very common. Even in today's world, such informal systems are not extinct. When formal banking systems fail to provide easy-to-access loans and sufficient returns on deposits or when there are no fully mature banking systems as typically seen in under-developed economies, private investors and small and medium-sized enterprises (SMEs) might seek alternatives. For example, there could be loans with less stringent conditions and deposits with more returns than in formal banking systems. One very well-known example of such alternatives is *microfinance*, which has sprung up to meet this need in many local communities around the world. It often refers to a formal or informal financial service that is to enhance the financial sustainability of the investors who lack access to formal banking services.

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Savings and loans are the two main services that microfinance can provide, and it also delivers other services such as money transfers and insurance, depending on service providers. Because such services are similar to the ones in formal banking systems and their analysis are well established in the existing research, we focus on one of the popular group-based models which incorporates savings and loans to satisfy participants' common interest, the so called *rotating savings and credit association* (ROSCA). Armendáriz and Morduch [2005] explained ROSCA as one of the roots of modern microfinance institutions. The basic framework of any ROSCA is as follows. A certain number of participants agree to make a regular meeting system with a fixed maturity. And at every meeting, each member puts in a fixed amount of money and the collected pot is then given to one of the members who has not yet received a pot. At the maturity of a ROSCA, i.e., when each member has received his/her pot exactly once, they either dissolve or restart the system.

Many SMEs use this system for business purposes, for example, to make a lump sum of money before the full amount is accumulated, to avoid transaction costs or taxes, and so on. Buckley [1997] mentioned that in Kenya, Malawi, and Ghana, ROSCA is a common source of enterprise finance and offers SMEs a self-sufficient, voluntary-based organizational framework to save and borrow money. Especially in Ghana, according to Owusu et al. [2013], most of the traders who lack access to funding consider ROSCA as the easiest and readily available alternative to raise funds to support their business operations. Similar cases can also be found in Asia. In Taiwan, for example, Gelinas [1998] stated, "Until 1970, the banks for small and medium-sized businesses in Taiwan were rotating savings and credit societies, clearly part of the informal sector. Recently integrated into the formal sector, these community-based banks still operate with *hehui* funds now totalling $3.8 billion."

In addition to being an effective funding source, ROSCA can be approached from the perspective of sustainable enterprises. As Anane et al. [2013] mentioned, microfinance products and services have economic impacts on SMEs: absorbing shocks and exposure, improving productivity, raising income or increasing savings. ROSCAs are no exception to this point. The paper by Khan and Lightfoot [2011] conducted a qualitative research on the sustainability of economic players supported by ROSCAs. To further highlight this point, Mbizi and Gwangwava [2013] investigated a similar research question in Zimbabwe. According to their research, ROSCAs help to smoothen business financial cycles, to manage cash flows, and to facilitate recapitalization of enterprises by pooling financial resources to one member per time period, thereby enhancing the operational sustainability of local enterprises.

Furthermore, it is intuitively appealing that ROSCAs play an important role in developing countries because they can replace some capital market functions [Scholten, 2000]. However, even in countries with big credit markets, some forms of ROSCAs still exist. For example, ROSCAs of the ordered type (hereafter, ordered ROSCA) in which there is a pre-determined order of recipients are very popular in South Korea. (See Scholten [2000] for another example in Germany and Austria.) We aim to better understand this ordered type ROSCA which coexists with mature formal financing systems. In particular, we are interested in the issue of ROSCA design mentioned in Besley et al. [1993] such as the rate of a ROSCA and the order of recipients. The central question we attempt to address is when a ROSCA can be beneficial compared to formal banking systems and we do so in both of the cases where the credit ratings of participating members are the same or different. To achieve these goals, we adopt the approach of replication strategies that are typically used in asset pricing theory. By comparing cash flows from a ROSCA with those from banking transactions, we explicitly define extra payoffs to ROSCA participants.

Our contributions to the literature are as follows. First, we analyze the ordered ROSCA which is an important type of ROSCA, but has received relatively less attention. And second, we provide prescriptions about when a ROSCA is actually good, what rate should be used, and how many members can be accommodated in the system, etc. Even though our analysis is on the ordered ROSCA, it is worth mentioning that the *ex post* analysis of a random ROSCA is the same as the analysis of the ordered one. This is because the contributions of a member in the random case can be transformed into those in the ordered case, by selecting a suitable rate once the time of receiving a pot is realized. Therefore, our analysis can be useful for understanding of random ROSCAs. Lastly, we take a more theoretical approach to the analysis of a famous form of informal microfinance system whereas the existing research in the Operations Research community by, for example, Gutiérrez-Nieto et al. [2007], Gutiérrez-Nieto et al. [2009], Amersdorffer et al. [2014], and Piot-Lepetit and Nzongang [2014] is based on statistical techniques such as data envelopment analysis. Thus, we hope that this work sheds some light on the design issue of financial products at microfinance institutions.

1.2 Background and related works

Microfinance operates both formally and informally, and so do ROSCAs. Formal ROSCAs run by commercial banks in Argentina, Ghana, and Mexico, for example, are explained in Schreiner [2000] and Vonderlack and Schreiner [2002]. They have long maturities, many members, and big pots. In this model, a bank pays interest to members who are yet to get the pot and receives interest from members who got the pot in the past. Also, the sustainability of a system is now the responsibility of banks (hence, fees are charged) and such products are under government regulations. Other examples include building and loan associations in Germany and Austria called *Bausparkassen* that account for 15% of loans in Germany, 45% of loans in Austria, and more than 20% of household deposits, according to Scholten [2000]. This model was adopted by other European countries such as Hungary and Poland in 1990's. On the other hand, informal ROSCAs do not have any guarantee of insurance in the event of a member's default, so they have shorter maturities, less members, and smaller pots than formal ROSCAs. In addition, interests are paid and received according to the rules made by members. Informal ROSCAs are more widely observed around the globe in various forms. Instead of listing all such practices, we briefly mention some notable examples from Asian and African countries where ROSCAs have long histories and they are still actively in use.

- *China and Taiwan* : China and Taiwan has a common history of 3,000 years of ROSCAs, called *hehui*. As reported in Li and Hsu [2009], there are several types of *hehui*. For example, *lunhui* is the ordered ROSCA, *yaohui* selects a recipient each time randomly (random ROSCA), and *biaohui* has a secret bidding procedure to choose a recipient at each meeting (bidding ROSCA). Especially in Taiwan, according to Levenson and Besley [1996], random and bidding ROSCAs are popular as a savings device and account for a large part of the informal financial sector, at least 20.5% and it might be as high as 85% according to another estimate.

- *Japan*: Dekle and Hamada [2000] studied ROSCAs in Japan, called *kō*, and they are mostly random and bidding types. This *kō* has been used for about 800 years. In the early 20th century, the *kō* was flourishing but the implicit rate of return was often so unfair that the Japanese Ministry of Finance provided a formula for the rate. After World War II, most *kō*s became so large that they formed regional mutual banks, but, finally converted into commercial banks. Nevertheless, small, informal *kō*s still operate in Japan.

- *Korea*: Campbell and Ahn [1962] described *kye*, the ROSCA in Korea, which is about 2,000 years old and yet quite popular today. They are mostly ordered and bidding types. Both forms usually contain interests which are accrued on regular contributions of a member of *kye* after she receives a pot.

- *Ghana and other West African countries*: According to Bortei-Doku and Aryeetey [1996], rotating *susu* clubs, the ROSCAs in Ghana have a high-profile due to the lack of proximity to banks in general, and many of them are founded by a need to overcome frequent shortages of cash in their business activities and in crisis situations. Bouman [1995] summarized ROSCAs in other West African countries such as Congo, Liberia, Ivory Coast, Togo, Nigeria, and Cameroon, where 50 to 95 percent of adults participate in ROSCAs.

- *Kenya*: Kimuyu [1999] conducted a survey in the Kenyan community which showed that 45% of those questioned were participating in ROSCAs. According to Anderson and Baland [2002] and Anderson et al. [2009], most of ROSCAs in Kenya are ordered ones, and only a few of them are random ROSCAs. They could not find any bidding ROSCAs in their surveys.

- *South Africa*: According to Burman and Lembete [1996], ROSCAs are known by various names in South Africa such as *stokvel*, *gooi-gooi*, *umgalelo*, *mahodisana*, and *umshayelwano*. In 1988, the National Stokvel Association of South Africa was established to claim the rights of *stokvel* members and promote recognition of *stokvels* by formal financial institutions as a source of informal credit.

In the large ROSCA literature, one key work was that of Besley et al. [1993] who provided an economic analysis of ROSCA, focusing on its economic role and performance. Their work was motivated by ROSCA practices observed in immigrants groups in the United States as well as in developing countries such as India. The model assumes that members of a ROSCA do not have an access to credit markets. The members have the objective of increasing their lifetime utilities dependent on an indivisible durable consumption good (a participant receives a constant flow of services for the rest of the lifetime upon purchase). Under some additional assumptions on utilities and preferences, they showed that ROSCAs (random or bidding) improve members' lifetime expected utilities compared to the case where individuals make savings by themselves. In addition to this result, discussed are the effect of heterogenous members, i.e., with different preferences, and the sustainability of ROSCA due to its informal nature.

More qualitative and quantitative investigations by many researchers followed. Bouman [1995] compared ROSCAs and Accumulating Savings and Credit Associations (ASCRAs) in which the savings are not instantly redistributed but allowed to accumulate to make loans. The author also proposed a hybrid of ROSCA and ASCRA where at each meeting, the highest bid is not redistributed but utilized as a loan fund. In the literature, the performances of ordered, random, and bidding types are often compared. To mention a few, Besley et al. [1994] studied allocations achieved by random and bidding types as well as those obtained by a credit market. A similar comparison was done by Kovsted and Lyk-Jensen [1999]. But in the latter, the authors developed a game theoretic model and assumed that members of a ROSCA can raise a fund outside the system at positive costs. Anderson et al. [2009] found that enforcement problems on defecting members are more severe in random ROSCAs than in ordered ones, and that the system is not sustainable without social sanctions regardless of whether it is random or ordered. Klonner [2003], on the other hand, extended existing models to incorporate risk-averse agents who might suffer from stochastic income shocks. Also, there is a model developed by Ambec and Treich [2007] which explains the existence of ROSCAs from the viewpoint of self-control problems.

In terms of finding conditions that would make ROSCAs more appealing than banks, our work shares the same spirit with van den Brink and Chavas [1997]. The authors looked at profits from ordered ROSCAs and from interest-bearing savings accounts but without any discounting nor interest compounding. Our research has four different aspects from their work. First, we utilize savings and loans together to replicate ROSCAs' cash flows, which helps us find profitable conditions of ROSCAs. Second, ROSCAs are found to be still feasible even when later positions receive more benefits via a quantitative analysis whereas van den Brink and Chavas [1997] noted that earlier positions are better in general. Third, we provide an answer to an optimal design problem of an ordered ROSCA with and without homogeneity of participants. Lastly, discounting and compounding calculations are employed in this paper.

The remainder of the paper is organized as follows. Section 2 introduces the model and formulates an optimization problem to derive an optimal design of a ROSCA. In the section that follows, we analyze the feasibility condition and solve for an optimal solution when members of the system are homogeneous in terms of credit ratings. In Section 4, we extend the analysis to the case of heterogeneous members, and Section 5 concludes. All proofs can be found in the appendix.

2. The Model

We begin our discussion by describing one form of ordered ROSCAs that is in widespread use. This informal finance system is assumed to have n *members*, say member 1 to member n, and *maturity* T at which it ceases to function. Throughout the paper, we assume that there are at least two members to avoid triviality, i.e., $n \geq 2$. In this system, members have regular meetings at times $\{T/n, 2T/n, \cdots, T\}$, so the total number of meetings is n. The important components of the system are its *interest rate* r and the order of members according to which exactly one member is entitled to receive a certain amount of cash at each meeting. We note that this rate determines returns to each member and it is different from banking deposit or loan rates. In practice, this rate is determined by an agreement among members before starting the rotation, depending on domestic economy, economic status of members, etc. Although the determinants of the rate have not been fully investigated in the literature, it is reported by Yu [2014] that the average interest rate in ROSCAs was around 0.5%. To be more specific about the system features, for each member, the amount of cash to put into the system at each time is different depending whether it is before she receives a pot or it is after. If it is before or at the time of the receipt, then she delivers a base amount, say 1, and if it is after, then $1 + nr$. If she is supposed to receive a pot at the k-th meeting, then the total amount of cash receipt is $\mathfrak{M}_k(r) := n + (k-1)nr$. This makes the total cash inflow to the system and the cash outflow equal at each meeting time. We provide the cash flow diagram of a member in the k-th position in Figure 1, where each integer i in circles means the i-th meeting, the numbers near down arrows are cash outflows, and the number near the up arrow is the cash inflow at the k-th meeting.

Besides the basic description of the model, we make some additional assumptions. First, we assume that all members abide by the contract details and do not default on their obligations. This is not entirely realistic because there were incidents such that one of the members ran away, making negative returns to members who had not received his/her pots. In this work, however, we focus on properties and design issues of the system, leaving the task of incorporating such default risk of a system as a future work. Second, different from the majority of the literature on ROSCAs, we assume that a mature banking system exists and that all members freely transact with banks. As it is typical in real world, a bank provides a single deposit rate r_d, e.g., three months CD rate, but requires different loan rates depending on the credit rating of each customer, say $r_l^{(i)}$ for member i. Throughout the paper, we assume that $0 \leq r_d \leq r_l^{(1)} \leq \cdots \leq r_l^{(n)}$ without loss of any generality. Here, deposit rate r_d and loan rate $r_l^{(i)}$ are the rates for each time period between the meetings. In all examples we consider in this paper, we take the interval between meetings equal to one month, and r_d and $r_l^{(i)}$ are monthly rates.

Next, let us consider the following hypothetical banking transactions. Throughout the first k meetings, member i makes a deposit of size 1 at each meeting. At the kth meeting, she withdraws all the savings of which size is denoted by $\mathfrak{D}_k$ and, additionally, takes a loan $\mathfrak{L}_k^{(i)}$ with rate $r_l^{(i)}$. This loan is paid back in full by the payment of $1 + nr$ for each of periods $k+1$ to n. The term nr means the simple interest of the loan for each period. Based on above assumptions, we observe that

$$\mathfrak{D}_k = \sum_{j=0}^{k-1} (1 + r_d)^j, \quad \mathfrak{L}_k^{(i)}(r) = \begin{cases} \sum_{j=1}^{n-k} (1 + nr)/(1 + r_l^{(i)})^j, & \text{if } k = 1, \cdots, n-1; \\ 0, & \text{if } k = n. \end{cases}$$

The cash flow diagram is given in Figure 2. Notice that in these transactions the cash outflows for member i are exactly the same as in the case of a ROSCA. Hence, the difference between the total cash inflows of banking transactions and a ROSCA can be understood as the extra profit by participating in a ROSCA. We write the extra profit of member i in the k-th position as follows:

$$\Pi_k^{(i)}(r) := \mathfrak{M}_k(r) - \mathfrak{D}_k - \mathfrak{L}_k^{(i)}(r), \quad i, k = 1, \ldots, n.$$

By looking at this difference, one realizes that the minimal condition for the existence of a ROSCA is that $\Pi_k^{(i)}(r)$ is nonnegative; otherwise, she would have earned a better return via banking transactions.

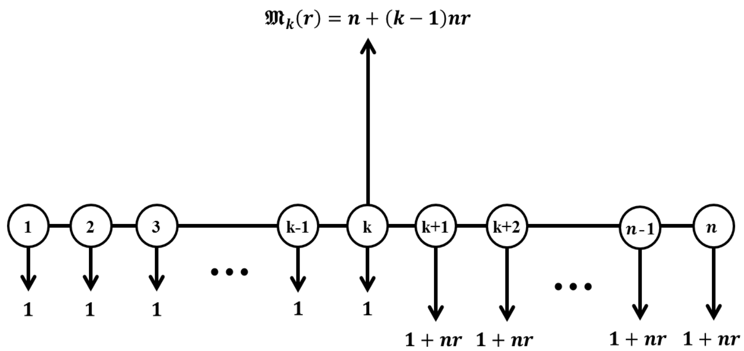

Figure 1: Cash flow diagram from a ROSCA of member i in the k-th position.

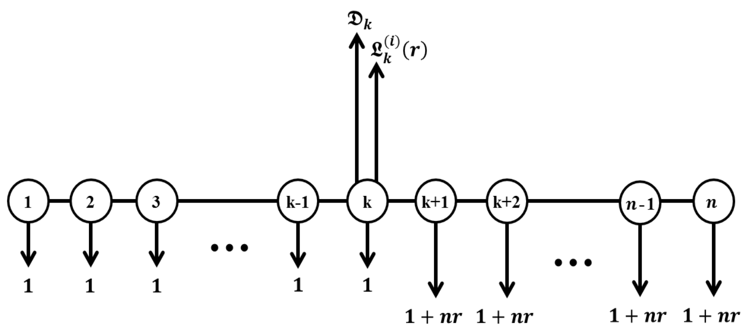

Figure 2: Cash flow diagram from banking transactions of member i in the k-th position.

Therefore, in the sections that follow, we conduct a feasibility analysis of the system by considering the condition $\Pi^{(i)}_{\sigma(i)}(r) \geq 0$ for all $i = 1, \ldots, n$ where σ represents the order of members, i.e., member i receives a pot at the $\sigma(i)$th meeting. Clearly, σ is a permutation of the set $\{1, \ldots, n\}$. Later in the paper, we compare different orders of members in terms of the sum of extra profits. For this purpose, we need a measure to compare extra profits at different meetings. We note that if a member makes an additional bank deposit $\pi^{(i)}_k(r) := \Pi^{(i)}_k(r)/(1 + r_d)^k$ at time 0, then this offsets her extra gain from a ROSCA at the kth meeting. We call this amount a discounted extra profit of member i in the k-th position . Hence, we use this as a tool to make comparisons between orders. One of our main questions, then, is the following optimization problem:

$$\begin{aligned} \max_{\sigma, r} \quad & \sum_{i=1}^{n} \pi^{(i)}_{\sigma(i)}(r) \\ \text{s.t.} \quad & r \geq 0,\ \pi^{(i)}_{\sigma(i)}(r) \geq 0, \quad i = 1, \ldots, n \end{aligned} \tag{1}$$

for given rates r_d and $r_l^{(i)}$'s. An optimal solution represents an optimal design of the informal finance system that guarantees better returns to the members than banking transactions and maximizes the total extra profits.

Remark 2.1 Let us consider random and ordered ROSCAs having the same interest rate r and the same number of participants n for their *ex-ante* comparison. Then, in a random case, member i's discounted extra profit is one of n possible values, $\pi_1^{(i)}(r), \pi_2^{(i)}(r), \dots, \pi_n^{(i)}(r)$, with probability $1/n$ each, whereas the discounted extra profit is $\pi_{\sigma(i)}^{(i)}(r)$ in an ordered case. The choice between the two types of ROSCAs might depend on this member's risk-taking characteristics. For instance, let us take the power utility function $u(x) = (x^{(1-\rho)} - 1)/(1-\rho)$. It can be shown that, for ρ sufficiently large (extremely risk averse), the ordered ROSCA gives a higher utility than the random one unless $\sigma(i) = \operatorname{argmin}_j \pi_j^{(i)}(r)$. On the other hand, in the case of a random ROSCA, the timing of the receipt of a pot may not necessarily meet the member's needs, which means a suboptimal use of funds. Along the same line, we lastly observe that an optimally ordered ROSCA always dominates in the sense of expected total profits, thanks to the formulation (1).

3. System Feasibility and Profitability

In this section, we assume that the members of a ROSCA are *homogeneous* in the sense that their loan rates are equal. This simplifies the analysis and allows us to avoid unnecessary complications in delivering our main messages. For the sake of notational convenience, we drop the superscript (i) from $r_l^{(i)}$, $\Pi_k^{(i)}$, $\pi_k^{(i)}$, and $\mathfrak{L}_k^{(i)}$. We are then concerned with the feasibility condition $\pi_k(r) \geq 0$ (or equivalently, $\Pi_k(r) \geq 0$) for all $k = 1, \dots, n$ in the problem (1). When it is possible to find a nonnegative real number r that satisfies the condition, we call it an *admissible* rate.

3.1 Benchmark case

A closer look at the extra profit $\Pi_k(r)$ reveals that it can be expressed by two auxiliary functions $f(\cdot)$ and $g(\cdot)$ where $\Pi_k(r) = f(k)r + g(k)$ and

$$f(k) = n\left[k - 1 - \sum_{j=1}^{n-k}(1+r_l)^{-j}\right] = n\left[k - 1 - \frac{1-(1+r_l)^{k-n}}{r_l}\right],$$

$$g(k) = n - \sum_{j=0}^{k-1}(1+r_d)^j - \sum_{j=1}^{n-k}(1+r_l)^{-j} = n - \frac{(1+r_d)^k - 1}{r_d} - \frac{1-(1+r_l)^{k-n}}{r_l}$$

for $k = 1, \dots, n$. Often we take $\mathbb{R}_+$ as the domain of $f(\cdot)$ and $g(\cdot)$, after completing summations. These two functions are monotone, $f(\cdot)$ increasing and $g(\cdot)$ decreasing. The following lemma is useful throughout this subsection. For a real number x, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer that is not less than x and the largest one that is not greater than x, respectively.

Lemma 3.1 *There exists a unique real number k_0 in the interval $(1, n)$ with $f(k_0) = 0$. And if $h(\cdot)$ is a real valued function such that $h(k) = -g(k)/f(k)$ for all $k = 1, \dots, n$ whenever $f(k)$ is nonzero, then $h(\cdot)$ is increasing on each of the sets $\{1, \cdots, \lceil k_0 - 1 \rceil\}$ and $\{\lfloor k_0 + 1 \rfloor, \cdots, n\}$.*

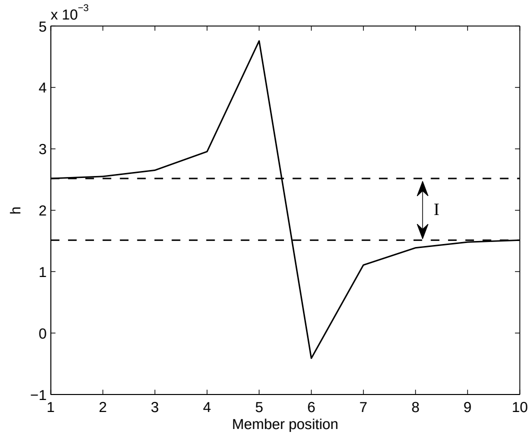

Figure 3: Graph of $h(\cdot)$ with respect to member position: $r_d = 0.3\%$, $r_l = 0.5\%$, and $n = 10$.

The lemma implies that $h(1) = \min_{k=1,\cdots,\lceil k_0-1 \rceil} h(k)$, and $h(n) = \max_{k=\lfloor k_0+1 \rfloor,\cdots,n} h(k)$. Now, we define a set

$$\mathcal{I} := [h(n), h(1)]$$

if the left end is not greater than the right end. Otherwise, $\mathcal{I}$ is set equal to the empty set. In addition, when $\mathcal{I}$ is not empty, it is a subset of $\mathbb{R}_+$ because $f(n) > 0$ and $g(n) \leq 0$, which can be easily checked. Figure 3 illustrates how $\mathcal{I}$ is constructed from $h(\cdot)$. It clearly shows $h(\cdot)$ is increasing up to $\lceil k_0 - 1 \rceil$ and increasing again from $\lfloor k_0 + 1 \rfloor$. In the figure, k_0 is located between 5 and 6. As long as $\mathcal{I}$ is nonempty, we can visualize the interval as the two dashed lines in the graph albeit the position of k_0 might be different. This interval plays a crucial role in describing the following result, which is the first main result of this paper.

Theorem 3.2 *The interval $\mathcal{I}$ is the set of all admissible rates. In addition, when $\mathcal{I}$ is nonempty, the optimality for the problem* (1) *is achieved at $r = h(1)$.*

Albeit simple, the above result completely solves our initial optimization problem (1) at least in the case of homogeneous members. This solution prescribes one way of utilizing the fact that there are typically positive interest rate margins, i.e., loan rate minus deposit rate, in a formal banking system. Indeed, if the interest rate margin is zero, then it is possible to show that there are severe restrictions on the choice of an admissible rate.

Corollary 3.3 *Suppose that the interest rate margin is zero, i.e., $r_d = r_l$. If the rates are zero or if there are only two members in the ROSCA, then we have $\mathcal{I} = \{r_d/2\} = \{r_l/2\}$; otherwise, $\mathcal{I} = \emptyset$.*

On the other hand, it is simple to verify that the equivalent statement for $\mathcal{I} \neq \emptyset$ is

$$\sum_{i=1}^{n-1} (1 + r_d)^i \cdot \sum_{j=1}^{n-1} \left(\frac{1}{1 + r_l} \right)^j \leq (n-1)^2. \tag{2}$$

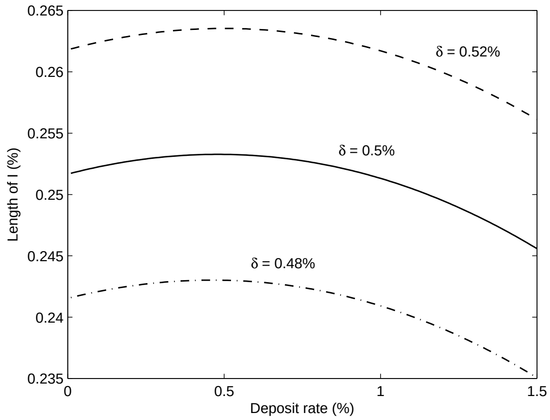

Figure 4: Length of $\mathcal{I}$ as a function of the deposit rate where fixing the interest rate margin δ at three different values when there are 10 members.

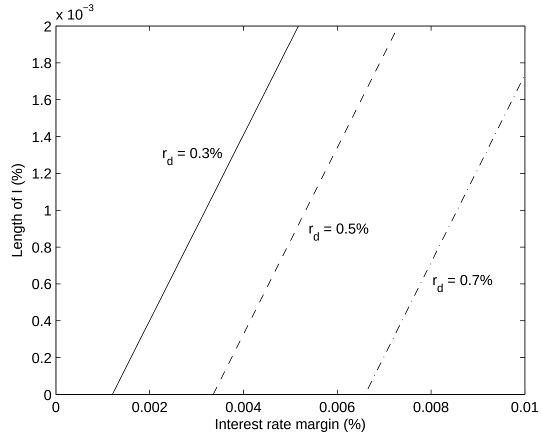

Figure 5: Length of $\mathcal{I}$ as a function of the interest rate margin with the deposit rate at 0.3%, 0.5%, and 0.7% when there are 10 members.

Note that $\mathcal{I}$ is a single point set when the equality holds. In addition, the left hand side strictly increases in r_d and strictly decreases in r_l. Hence, for each fixed r_d (or r_l), there is a unique r_l (or r_d) at which $\mathcal{I}$ collapses to a single point. We omit detailed computations because they are straightforward, but record one consequence as a corollary.

Corollary 3.4 *For each fixed r_d, there is a threshold value for r_l such that $\mathcal{I}$ has zero length for the first time, i.e. when the equality holds in* (2). *A similar statement holds for r_l. Consequently, if the interest rate margin is sufficiently large, then there exists an admissible rate.*

The importance of this observation lies in that it provides one possible reason why ROSCAs are still popular in countries with developed finance systems. In the literature, ROSCAs are usually described as a method of accumulating cash amounts for purchase of goods in under-developed economies. Actually, it was discussed by, e.g., Callier [1990] that they will disappear as capital market integration progresses because lumpy expenditures are possible due to the accessibility to credits. However, it is not difficult to find out that the informal finance systems are still popular among citizens in countries with big credit markets, e.g., Japan, South Korea, and Taiwan. Easy access and no tax can be one reason for such popularity, but the above analysis shows that it is actually possible to make better returns when the interest rate margins are not favorable. One related observation is that the threshold is usually formed when the interest rate margin is large enough. This essentially follows from the difference between the banking rates and the return of the ROSCA (no compound interests). Hence, having nonzero interest rate margin is not enough to construct a profitable ROSCA.[1]

[1] The World Bank reports the interest rate spread, defined as the interest rate charged by banks on loans to private sector customers minus the interest rate paid by commercial or similar banks for demand, time, or savings deposits, in Japan every year, and Google Trends provides search query time series on rotating savings and credit association in Japan. If this Google Trends result is interpreted as the expression of interests on ROSCAs in Japan, then this gives an indirect but empirical relationship between the interest rate margin and the demand on ROSCAs. The correlation between those two time series from 2005 to 2014 is 0.8932.

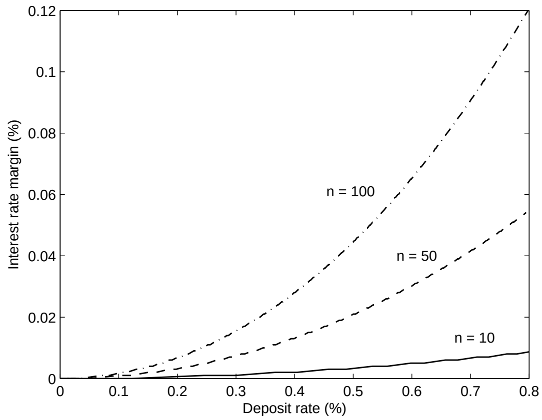

Figure 6: The smallest interest rate margin such that $|\mathcal{I}|$ becomes zero where n is the number of members in the system.

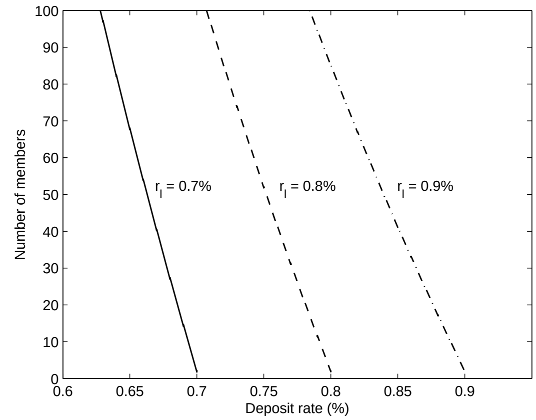

Figure 7: The maximal number of members that produces a feasible solution to (1) as a function of the deposit rate.

Although it is tempting to conclude that there are more opportunities to exploit if the interest rate margin increases, the relationship is not that simple because $|\mathcal{I}|$ depends on r_d or r_l as well as the margin itself. One such example is given in Figures 4 and 5. Figure 4 shows the graphs of the interval $\mathcal{I}$ as a function of r_d when the interest rate margin is fixed at $\delta > 0$. Figure 5 contains similar information but graphs are illustrated as a function of the interest rate margin δ with three different levels of r_d. The former shows that overall $|\mathcal{I}|$ tends to increase as the margin δ does so, but its size changes nonlinearly as r_d varies, while the latter illustrates that for each deposit rate, there is a threshold level for the interest rate margin satisfying $|\mathcal{I}| = 0$ beyond which $\mathcal{I}$ becomes a nontrivial interval and there are admissible rates.

A more interesting relationship is exhibited in Figure 6. It describes graphically the minimal interest rate margin (as a function of r_d) above which $|\mathcal{I}|$ becomes strictly positive so as to make a ROSCA profitable. It also shows how those graphs behave as the number of participants increases. Again, all rates are computed on a monthly basis. Basically, it tells us that we need higher interest rate margins to get a ROSCA system going if there are more members in the system or the deposit rate gets better. Looking at this latter observation from a different point of view, one realizes that there should be a certain threshold for the maximal number of members in a ROSCA if all other model parameters are fixed at constants. In fact, the left end of $\mathcal{I}$ can be shown to diverge to infinity as n increases while the right end converges to r_l. This is illustrated in Figure 7. The graph shows the maximal number of participants that can be accommodated in a given ROSCA as a function of r_d at three different levels of r_l. One can clearly see that a ROSCA can have a larger number of members when the interest margin becomes larger.

3.2 Additional issues

Let us turn our attention to the original question (1). Even though it is clear that members can benefit from participating in a ROSCA in certain situations, the nonnegativity of extra profits might not be the only constraint. As mentioned earlier, there exists the possibility of a member reneging on the ROSCA rules and there are examples in which such incidents caused social concerns, leading to regulatory problems as well as increased business bankruptcies and corporate debts. Also, there are some research works on such behaviors and the design of ROSCAs to prevent them. For instance, Wei and Lijuan [2011] proposed the legal regulation of ROSCAs applying partnership to those systems. For further issues, we refer the interested reader to Besley et al. [1993] and Handa and Kirton [1999]. In addition, a member might lose investment opportunities if the timing of the receipt of a pot is late. Due to these issues of sustainability and opportunity costs, it is plausible that members in later positions would want additional compensations. Apart from these concerns, it is also reasonable to imagine that members are also likely to compare their extra profits to each other. This is a fairness issue which often becomes quite subtle. We argue below that this problem cannot be resolved completely, but still a partial answer is presented.

Hence, to deal with the first problem at least partially, we impose the following constraint:

$$\pi_k(r) \leq \pi_{k+1}(r), \quad k = 1, \ldots, n-1. \tag{3}$$

In other words, more extra profits to members who receive pots later. It is easy to see that this condition is transformed to $\pi_{k+1}(r) - \pi_k(r) = \Delta_f(k)\, r + \Delta_g(k) \geq 0$ for all $k = 1, \ldots, n-1$ where

$$\Delta_f(k) := \frac{f(k+1)}{(1+r_d)^{k+1}} - \frac{f(k)}{(1+r_d)^k}, \quad \Delta_g(k) := \frac{g(k+1)}{(1+r_d)^{k+1}} - \frac{g(k)}{(1+r_d)^k}.$$

Therefore, the optimization problem (1) with the additional constraint (3) becomes an easily implementable linear programming problem. The next result provides a feasibility analysis and a solution for this question.

Proposition 3.5 *There exists a nonnegative rate r satisfying* (3) *if and only if the following condition holds:*

$$(r_l - r_d) < \frac{r_l(2-(k-1)r_d)}{1-(1+r_l)^{-n+k}}, \quad k = 1, \ldots, n-1.$$

In this case, the feasible region for the problem (1) *plus* (3) *is* $\mathcal{I}' := \mathcal{I} \cap \left\{ r : r \geq \max_k \{-\Delta_g(k)/\Delta_f(k)\} \right\}$, *and an optimal solution is obtained at the right endpoint of* $\mathcal{I}'$.

Simple sufficient conditions for (3) can be found. For example, the condition in the proposition is satisfied if $(r_l - r_d) < r_l(2-(k-1)r_d)$ for all k, which is equivalent to $(n-2) < 1/r_d + 1/r_l$. This expression shows the relationship between n, r_d, and r_l to guarantee the increasing returns to ROSCA members more clearly; too many members or sufficiently large deposit/loan rates make the inequality invalid. Actually, the parameter values in our numerical examples are small enough to satisfy (3), but it is numerically verified that the inequality in Proposition 3.5 is violated for large n, r_d, or r_l. Hence, a large interest rate margin is not enough to meet the constraint on the increasing property of discounted extra profits.

As for the second issue of fairness, one easily can see that there is no general functional form of rate r in terms of r_d, r_l which yields the same extra profit for all members. This is because $\mathfrak{M}_k(r)$ is linear in r

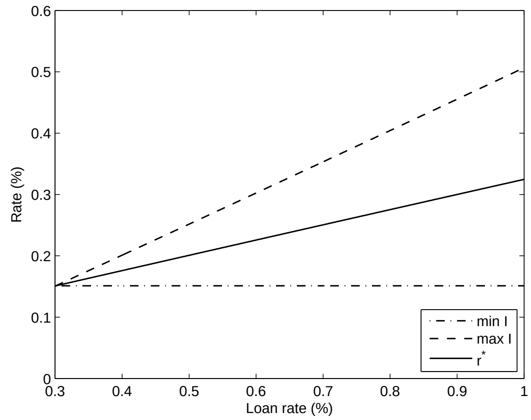

Figure 8: $\min \mathcal{I}$, $\max \mathcal{I}$, and r^* in terms of r_l when $r_d = 0.3\%$ and $n = 10$.

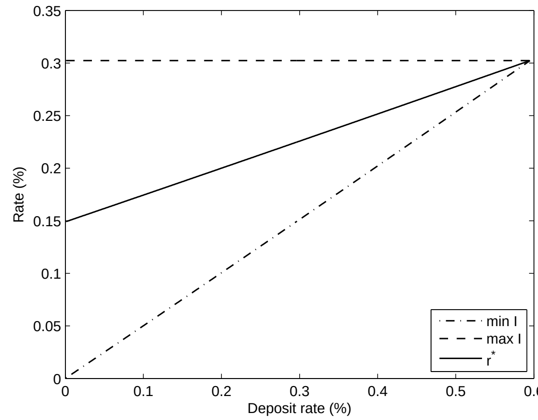

Figure 9: $\min \mathcal{I}$, $\max \mathcal{I}$, and r^* in terms of r_d when $r_l = 0.6\%$ and $n = 10$.

but $\mathfrak{D}_k$ and $\mathfrak{L}_k(r)$ are higher-order polynomials in r_d and $1/(1 + r_l)$. Instead of such an extreme case, we consider an approach of imposing a bound on the mean squared deviation of extra profits:

$$\sum_{k=1}^{n} \Big(\pi_k(r) - \overline{\pi}(r)\Big)^2 \leq M \tag{4}$$

where $\overline{\pi}(r) = n^{-1} \sum_{k=1}^{n} \pi_k(r)$ and M is some positive real number. The left side of the inequality is a quadratic function of r, say $ar^2 + 2br + c$ with positive a. The next result parallels Proposition 3.5. We omit its proof which is almost trivial.

Proposition 3.6 *There exists a nonnegative rate r satisfying* (4) *if and only if $M \geq c - b^2/a$. In this case, the feasible region for the problem* (1) *plus* (4) *is $\mathcal{I}'' := \mathcal{I} \cap \Big\{r : ar^2 + 2br + c \leq M\Big\}$, and an optimal solution is obtained at the right endpoint of $\mathcal{I}''$.*

Remark 3.7 The best possible r to achieve fairness in the sense that we explained above, if admissible, is the rate r^* that minimizes the mean squared deviation in (4). Clearly, $r^* = -b/a$. Note that r^* is different from the optimal solution in Proposition 3.6. Figures 8 and 9 are two examples from which we see that r^* is located around the middle of $\mathcal{I}$.

3.3 Defaultable case

In this subsection, we assume that the system can break down due to the default of a member or some exogenous causes. All other assumptions remain the same as in the benchmark case. For default probabilities, we adopt the constant hazard rate model which is based on exponential failure distribution, assuming that defaults happen between the meetings. Let $Q_t = \mathsf{P}(t < \tau < t+1)$, $t = 1, \ldots, n-1$, denote the probability that a default occurs at $\tau \in (t, t+1)$. When $t = n$, $Q_n = \mathsf{P}(\tau > n)$ is the probability of no default. Then, the constant hazard rate leads to $Q_t = e^{-\lambda t}(e^{\lambda} - 1)$, $t = 1, \ldots, n-1$, and $Q_n = e^{-\lambda(n-1)}$. This type of approach has been adopted in other contexts as well. For example, Banasik et al. [1999] applied survival analysis techniques to study credit scoring systems.

We consider the same hypothetical banking transactions as in Section 2. The difference between the benchmark case and the defaultable case occurs when $k > t := \lfloor \tau \rfloor$, i.e., a ROSCA fails before the member in k-th position receives the pot. In this case, by ceasing to make deposits at a bank and withdrawing all the savings at k, we can match the cash outflows in a ROSCA and banking transactions. However, the cash inflow of a ROSCA is zero while the banking transaction gives $\mathfrak{D}_t(1 + r_d)^{k-t}$. Hence, the extra inflow is $-\mathfrak{D}_t(1 + r_d)^{k-t}$.

On the other hand, if $k \leq t$, we note that there is no cash outflow for a ROSCA at $t+1, \ldots, n$, while the member is obliged to pay bank $1 + nr$ in each period in the banking transactions. This makes the extra profit in the defaultable case is no less than the extra profit $\Pi_k(r) = \mathfrak{M}_k(r) - \mathfrak{D}_k - \mathfrak{L}_k(r)$ in the benchmark case. Consequently, for example, the condition $\Pi_k(r) \geq 0$ becomes a sufficient condition that guarantees the profitability of a ROSCA in this particular situation. Then, the outflow of the ROSCA is not bigger than that of the banking transaction.

Combining these two separate considerations, we realize that one sufficient condition for the profitability of a ROSCA in the defaultable case is that the expected extra inflow is nonnegative, i.e.,

$$\bar{\Pi}_k(r) := \sum_{t=1}^{n} \Pi_{k,t}(r) \cdot Q_t \geq 0$$

where

$$\Pi_{k,t}(r) := \begin{cases} -\mathfrak{D}_t(1 + r_d)^{k-t}, & \text{if } k > t; \\ \mathfrak{M}_k(r) - \mathfrak{D}_k - \mathfrak{L}_k(r), & \text{otherwise,} \end{cases}$$

for $k = 1, \ldots, n$. We call $\bar{\pi}_k(r) := \bar{\Pi}_k(r)/(1 + r_d)^k$ the discounted expected extra inflow of the member in the k-th position . Now, we can proceed similarly as in Section 3.1, the main question being the optimization problem

$$\begin{aligned} \max_{r} \quad & \sum_{k=1}^{n} \bar{\pi}_k(r) \\ \text{s.t.} \quad & r \geq 0, \ \bar{\pi}_k(r) \geq 0, \quad k = 1, \ldots, n \end{aligned}$$

for given r_d, r_l, and λ.

The expected extra inflow $\bar{\Pi}_k(r)$ can also be expressed by two auxiliary functions $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$ where $\bar{\Pi}_k(r) = \bar{f}(k)r + \bar{g}(k)$ and

$$\bar{f}(k) = ne^{-\lambda(k-1)} \left[k - 1 - \sum_{j=1}^{n-k} (1 + r_l)^{-j} \right],$$

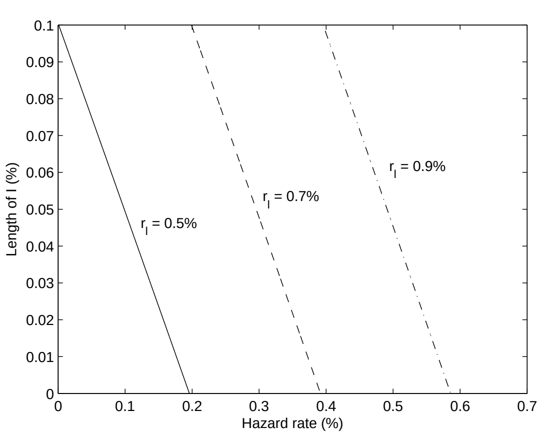

Figure 10: $|\bar{\mathcal{I}}|$ as a function of λ with three different r_l, and $n = 10$, $r_d = 0.3\%$.

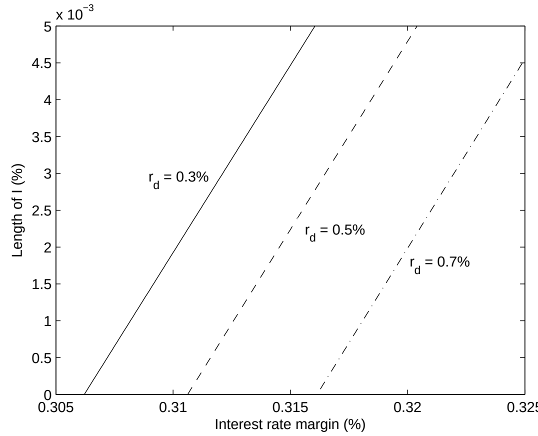

Figure 11: $|\bar{\mathcal{I}}|$ as a function of $r_l - r_d$ with three different r_d, and $n = 10$, $\lambda = 0.3\%$

$$\bar{g}(k) = e^{-\lambda(k-1)} \left[n - \sum_{j=0}^{k-1} \left\{ e^{\lambda}(1 + r_d) \right\}^j - \sum_{j=1}^{n-k} (1 + r_l)^{-j} \right],$$

for $k = 1, \ldots, n$. As before, we complete the summations and use $\mathbb{R}_+$ as the domain of $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$. The next lemma is helpful in deriving the following corollaries which provide consistent results with Section 3.1 as long as the hazard rate is small enough. For notational simplicity, we set $\lambda^* = \log\{(1 + r_l)/(1 + r_d)\}$.

Lemma 3.8 *There exists a unique real number k_0 in the interval $(1, n)$ with $\bar{f}(k_0) = 0$. In addition, if $\lambda \le \lambda^*$ and $\bar{h}(\cdot)$ is a real valued function such that $\bar{h}(k) = -\bar{g}(k)/\bar{f}(k)$ for all $k = 1, \ldots, n$ whenever $\bar{f}(k)$ is nonzero, then $\bar{h}(\cdot)$ is increasing on each of the sets $\{1, \cdots, \lceil k_0 - 1 \rceil\}$ and $\{\lfloor k_0 + 1 \rfloor, \cdots, n\}$.*

A similar implication as in Section 3.1 can be made from this lemma. Let us consider, for any λ in the interval $[0, \lambda^*]$, we set $\bar{h}(1) = \min_{k=1,\cdots,\lceil k_0 - 1 \rceil} \bar{h}(k)$, and $\bar{h}(n) = \max_{k=\lfloor k_0+1 \rfloor,\cdots,n} \bar{h}(k)$. We then define an interval

$$\bar{\mathcal{I}} := \left[\bar{h}(n), \bar{h}(1)\right]$$

if $\bar{h}(n) \le \bar{h}(1)$; $\bar{\mathcal{I}} = \emptyset$ otherwise. It can be easily checked that $\bar{\mathcal{I}} \subset \mathbb{R}_+$ because $\bar{h}(n) \ge 0$. Additionally, one can verify that $\lambda > \lambda^*$ implies $\bar{\mathcal{I}} = \emptyset$. This means that the sufficient condition for the profitability of a ROSCA does not hold if the default rate is large, which concurs with intuition. Consequently, we shall impose this constraint for the rest of our analysis. The next result is the analogue of Theorem 3.2 for the defaultable case. Its proof is quite similar except that we replace $(1 + r_d)$ with $e^{\lambda}(1 + r_d)$, hence omitted.

Theorem 3.9 *Assume that $\lambda \le \lambda^*$. The interval $\bar{\mathcal{I}}$ is the set of all admissible rates. In addition, when $\bar{\mathcal{I}}$ is nonempty, the optimality for the problem* (1) *is achieved at $r = \bar{h}(1)$.*

From the results so far, we see that zero interest rate margin implies $\lambda^* = 0$, thus the profitability of a defaultable ROSCA cannot be guaranteed. In addition, it is not difficult to show that the length of $\bar{\mathcal{I}}$ is a decreasing function of λ, r_d and an increasing function of r_l. In fact, for fixed r_d (r_l), there is a threshold for r_l (r_d) beyond which $\bar{\mathcal{I}}$ becomes nonempty as long as $\lambda \leq \lambda^*$. Also for fixed λ, we can find an admissible rate if the interest rate margin is large enough, which is a similar conclusion to Corollary 3.4.

The relationship between the hazard rate and the length of $\bar{\mathcal{I}}$ is illustrated in Figure 10 where $|\bar{\mathcal{I}}|$ is drawn as a function of λ. As the intuition suggests, the length decreases as there is a higher chance of defaults and the interest rate margin is lower. Figure 11 shows a similar result as in Figure 5, graphs of $|\bar{\mathcal{I}}|$ in terms of the interest rate margin. However, the length is smaller with $\lambda = 0.3\%$ compared to $\lambda = 0$ case.

In the next section, we conduct the basic analysis and compare the performances of three optimization problems in a more general setting, i.e., the original optimization problem (1), the same problem with the additional constraint (3), and lastly the original problem with (4). We also consider the defaultable case of ROSCA for heterogeneous members.

4. Optimal Ordering and System Efficiency

As opposed to the previous section where we considered the case of *homogeneous* members, we are concerned with *heterogeneous* members in this section. Recall that member i is assumed to have the loan rate $r_l^{(i)}$ and the loan rates are ordered as $0 \leq r_d \leq r_l^{(1)} \leq \cdots \leq r_l^{(n)}$. The critical difference is that we have order dependent constraints $\pi_{\sigma(i)}^{(i)}(r) \geq 0$ for all i. From now on, we call such a nonnegative rate σ-*admissible*.

4.1 Benchmark case

Before presenting our solution approach, we introduce some notation for convenience. First, we set

$$-\pi_k^{(i)}(r) = -\Pi_k^{(i)}(r)/(1+r_d)^k =: \alpha_{ik} r + \beta_{ik}$$

where α_{ik} and β_{ik} are values dependent on r_d and $r_l^{(i)}$ only. Second, for each $i, k \in \{1, \ldots, n\}$, we use x_{ik} for a variable with values 0 or 1. Since any permutation σ is a bijection from $\{1, \ldots, n\}$ onto itself, we can rephrase (1) as the following mixed integer nonlinear programming:

Problem Q_0

$$\begin{aligned}
\min_{x_{ik}, r} \quad & \sum_{i,k=1}^{n} x_{ik}\left(\alpha_{ik} r + \beta_{ik}\right) \\
\text{s.t.} \quad & r \geq 0, \\
& x_{ik}\left(\alpha_{ik} r + \beta_{ik}\right) \leq 0, \quad i, k = 1, \ldots, n, \\
& \sum_{i=1}^{n} x_{ik} = 1, \quad k = 1, \ldots, n,
\end{aligned}$$

$$\sum_{k=1}^{n} x_{ik} = 1, \quad i = 1, \ldots, n,$$
$$x_{ik} \in \{0, 1\}, \quad i, k = 1, \ldots, n.$$

This is exactly Problem Q_0 in Oral and Kettani [1992] except for that there are quadratic constraints in our formulation. However, it turns out that they do not add extra complexity because those terms also appear in the objective function. We follow the approach taken in Oral and Kettani [1992] and, for this, we need to find upper and lower bounds for $\alpha_{ik} r + \beta_{ik}$ for each i, k.

Going back to the original problem (1), suppose that we have a feasible (σ, r) such that $\pi^{(i)}_{\sigma(i)}(r) \geq 0$ for all $i = 1, \ldots, n$. Then, there exists some i with $\sigma(i) = 1$ because σ is invertible. Thus, $\alpha_{i1} r + \beta_{i1} \leq 0$. But, we note

$$\alpha_{i1} = \frac{n \sum_{j=1}^{n-1} (1 + r_l^{(i)})^{-j}}{1 + r_d} > 0 \Rightarrow r \leq -\frac{\beta_{i1}}{\alpha_{i1}}.$$

Therefore, for any feasible (σ, r), we get

$$0 \leq r \leq -\min_{i=1,\ldots,n} \frac{\beta_{i1}}{\alpha_{i1}} =: \overline{r}.$$

On this compact interval $[0, \overline{r}]$, linear functions $\alpha_{ik} r + \beta_{ik}$ obtain global min and max and we write D^-_{ik} and D^+_{ik}, respectively, following the notation in Oral and Kettani [1992]. Then, based on Proposition 1 in their paper, a mixed integer programming (MIP) appears:

Problem Q_1

$$\begin{aligned}
\min_{x_{ik}, r, \zeta_{ik}} \quad & \sum_{i,k=1}^{n} \left(D^-_{ik} x_{ik} + \zeta_{ik} \right) \\
\text{s.t.} \quad & r \geq 0, \\
& D^-_{ik} x_{ik} + \zeta_{ik} \leq 0, \quad i, k = 1, \ldots, n, \\
& \zeta_{ik} \geq \alpha_{ik} r + \beta_{ik} - D^-_{ik} x_{ik} - D^+_{ik}(1 - x_{ik}), \quad i, k = 1, \ldots, n, \\
& \zeta_{ik} \geq 0, \quad i, k = 1, \ldots, n, \\
& \sum_{i=1}^{n} x_{ik} = 1, \quad k = 1, \ldots, n, \\
& \sum_{k=1}^{n} x_{ik} = 1, \quad i = 1, \ldots, n, \\
& x_{ik} \in \{0, 1\}, \quad i, k = 1, \ldots, n.
\end{aligned}$$

Looking closely at the constraints, if $x_{ik} = 0$, then the constraints on ζ_{ik} reduce to $\zeta_{ik} = 0$. If $x_{ik} = 1$, then it is $0 \geq D^-_{ik} + \zeta_{ik} \geq \alpha_{ik} r + \beta_{ik}$. Since the coefficient of $D^-_{ik} x_{ik} + \zeta_{ik}$ in the objective function is 1, it is never optimal to have $D^-_{ik} + \zeta_{ik} > \alpha_{ik} r + \beta_{ik}$ when $x_{ik} = 1$. This reasoning shows that two problems are equivalent. See Oral and Kettani [1992] for more details. Hence, we can solve (1) using MIP solvers as long as r_d and $r_l^{(i)}$'s make the problem feasible.

Although the above formulation would yield an optimal design of a ROSCA, it is not analytically tractable and does not provide insights. Hence, we choose to work on a subset of the feasible region for the rest of this section. First, we define $\mathcal{I}^{(i)}$ to be the set of admissible rates for the ROSCA with homogeneous members that have deposit rate r_d and loan rate $r_l^{(i)}$. Then, it is not difficult to check that $\mathcal{I}^{(1)} \subseteq \cdots \subseteq \mathcal{I}^{(n)}$. Indeed, the right endpoint of $\mathcal{I}^{(i)}$ is, if non-empty,

$$\frac{n-1-\sum_{j=1}^{n-1}(1+r_l^{(i)})^{-j}}{n\sum_{j=1}^{n-1}(1+r_l^{(i)})^{-j}} = \frac{n-1}{n\sum_{j=1}^{n-1}(1+r_l^{(i)})^{-j}} - \frac{1}{n},$$

which is increasing in i thanks to $r_l^{(1)} \leq \cdots \leq r_l^{(n)}$. And the left endpoint of $\mathcal{I}^{(i)}$ is independent of $r_l^{(i)}$. As a result, if $r \in \mathcal{I}^{(1)}$, then it is σ-admissible for any order σ. A new optimization problem, as a second step, is formulated as follows:

$$\max_{\sigma, r\in\mathcal{I}^{(1)}} \sum_{i=1}^{n} \pi_{\sigma(i)}^{(i)}(r). \tag{5}$$

We emphasize that this problem is not a mere mathematical exercise. The nonnegativity of extra profits works as an incentive for a member to participate in a ROSCA. It is conceivable that since a member might not know her position *ex ante*, she wants to make sure that she gets a nonnegative extra profit regardless of her position. Hence, having a σ-admissible rate r for any order can be considered as a reasonable incentive scheme in a ROSCA with heterogenous members. The following theorem parallels Theorem 3.2.

Theorem 4.1 *Suppose that* (5) *is feasible. Then, the optimality for the problem is achieved at the right endpoint of* $\mathcal{I}^{(1)}$ *with an optimal order* $\sigma(i) = n+1-i$ *for* $i = 1, \ldots, n$.

Remark 4.2 Voorneveld [2003] argued that, in the discussion of optimality, one plausible alternative is the notion of *Pareto optimality*. This type of approach is quite common as we see, for example, in several studies using Pareto optimality in multi-objective optimization problem such as by Shulkla and Deb [2007], Warburton [1987], Yano and Sakawa [1989]. We say that σ is Pareto optimal if there is no other order which results in at least one individual having the discounted extra profit better off with no individual having it worse off, i.e., no $\widetilde{\sigma}$ such that for all $i = 1, \ldots, n$, $\pi_{\sigma(i)}^{(i)}(r) \leq \pi_{\widetilde{\sigma}(i)}^{(i)}(r)$ with at least one of inequalities being strict. It is obvious that the optimal order in Theorem 4.1 is Pareto optimal; otherwise, it couldn't have been optimal in the first place. In the case of a ROSCA with homogeneous members, every order is Pareto optimal. However, the result is quite different in the heterogeneous case. As an illustration, Figures 12 and 13 exhibit the efficiencies of all possible orders sorted from the lowest to the highest with a Pareto optimal order marked with a square. Here, the efficiency of an order σ is defined by

$$\text{efficiency}(\sigma) = \frac{\sum_{i=1}^{n} \pi_{\sigma(i)}^{(i)}(r)}{\max_{\widetilde{\sigma}\in\mathcal{S}} \sum_{j=1}^{n} \pi_{\widetilde{\sigma}(j)}^{(j)}(r)}$$

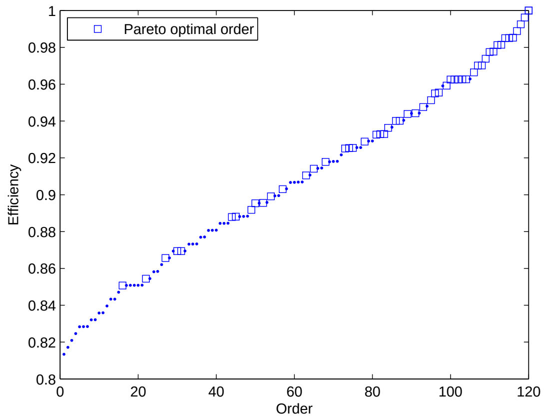

Figure 12: Efficiencies of orders when $n = 5$, $r = 0.3\%$, $r_d = 0.2\%$, and $r_l^{(i)}$'s are equally spaced between 0.7% and 0.9%.

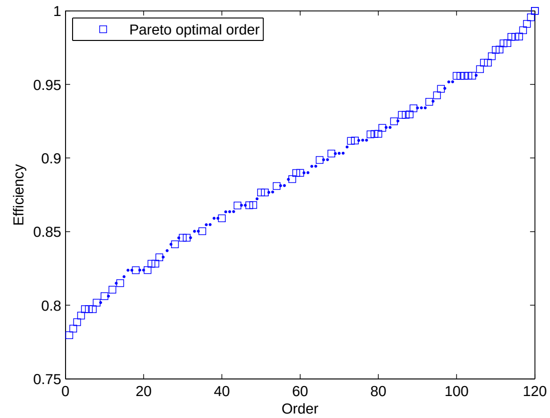

Figure 13: Efficiencies of orders when $n = 5$, $r = 0.4\%$, $r_d = 0.4\%$, and $r_l^{(i)}$'s are equally spaced between 0.8% and 1.0%.

where $\mathcal{S}$ is the set of every permutation of $\{1, \ldots, n\}$. One also notices that, depending on parameters, Pareto optimal orders can be observed in different regions of $[0, 1]$. However, we can prove some sufficient conditions that guarantee the Pareto optimality of every order. The reader is referred to the appendix for a proof.

Proposition 4.3 *Every order is Pareto optimal if $\pi_k^{(n)}(r)$ is strictly increasing in k, or if $\pi_k^{(1)}(r)$ is strictly decreasing in k.*

Remark 4.4 We note that there have been continued interests in the efficiency of microfinance institutions as we briefly mentioned in the introduction. For instance, Gutiérrez-Nieto et al. [2007] studied the financial efficiency of microfinance institutions, and Gutiérrez-Nieto et al. [2009] extended their previous model to measure both financial and social efficiency. Amersdorffer et al. [2014] assessed financial and social performance of credit cooperatives in Bulgaria, and Piot-Lepetit and Nzongang [2014] considered the relationship between financial sustainability and poverty outreach within microfinance institutions in Cameroon. These are empirical studies using data envelopment analysis, and their efficiencies are calculated to compare one microfinance institution with another. The efficiency in this paper takes a different point of view as it aims at understanding analytically how much the total discounted extra profit is reduced with non-optimal orders applied.

One interpretation of Theorem 4.1 is that ROSCAs can be beneficial for members who are less financially stable and thus possibly in greater needs of cash by receiving pots at earlier times while their cash-flow structures are socially optimal at the same time. As a final remark, we can also show that if $r_l^{(1)} < \cdots < r_l^{(n)}$, i.e., there is no pair of members with the same credit rating, then the order in the theorem is the unique optimal order that achieves the maximal efficiency.

4.2 Additional issues

In this subsection, we study two additional constraints and the resulting optimal solutions as in Section 3. The first one is the constraint by which a ROSCA compensates opportunity costs caused by receiving pots at later times. Similarly as in the previous section, it can be expressed as

$$\pi_k^{(i)}(r) \le \pi_{k+1}^{(i)}(r), \quad k = 1, \dots, n-1 \tag{6}$$

for each member i. However, in the proof of Proposition 4.3, it was shown that if $\pi_k^{(n)}(r)$ is increasing in k, then so is $\pi_k^{(i)}(r)$ for all i. Thus, (6) is equivalent to $\pi_k^{(n)}(r) \le \pi_{k+1}^{(n)}(r)$ for $k = 1, \dots, n-1$. Then, we are in the exact same position as in the case of homogeneous members except that r_l is replaced with $r_l^{(n)}$. Hence, the same arguments in the proof of Proposition 3.5 lead us to the proposition below. Regarding an optimal order, we note that the constraint does not affect the order of members, hence the optimal order in Theorem 4.1 remains unchanged. The obvious analogues of $\Delta_f(k)$ and $\Delta_g(k)$ are $\Delta_f^{(n)}(k)$ and $\Delta_g^{(n)}(k)$.

Proposition 4.5 *There exists a nonnegative rate r satisfying* (6) *if and only if the following condition holds:*

$$\left(r_l^{(n)} - r_d\right) < \frac{r_l^{(n)}(2-(k-1)r_d)}{1-\left(1+r_l^{(n)}\right)^{-n+k}}, \quad k = 1, \dots, n-1.$$

In this case, the feasible region for the problem (5) *plus* (6) *is* $\mathcal{I}^{(1)'} := \mathcal{I}^{(1)} \cap \Big\{r : r \ge \max_k \Big\{-\Delta_g^{(n)}(k)/\Delta_f^{(n)}(k)\Big\}\Big\}$*, and an optimal solution is obtained at the right endpoint of* $\mathcal{I}^{(1)'}$ *with* $\sigma(i) = n+1-i$.

From above, we realize that the spectrum of loan rates is one meaningful factor which determines the feasibility and optimality of a ROSCA. The loan rate and the interest rate margin for member 1 set the scene for a ROSCA, but the feasible region can be significantly restricted depending on the loan rate and the interest rate margin for member n. One sufficient condition for the feasibility is that $r_l^{(1)} - r_d$ is large enough to make $\mathcal{I}^{(1)}$ nonempty and plus that $(n-2) < 1/r_d + 1/r_l^{(n)}$ is also satisfied. This second condition gives us a clear picture about the relationship between parameter values that guarantee the increasing returns to ROSCA members. Similarly to what we observed in Section 3.2, too many members or large deposit/loan rates make the above inequality violated.

Now, let us move onto the other constraint which bounds the mean squared deviation of the discounted extra profits:

$$\sum_{i=1}^{n}\left(\pi_{\sigma(i)}^{(i)}(r) - \frac{1}{n}\sum_{j=1}^{n}\pi_{\sigma(j)}^{(j)}(r)\right)^2 \le M \tag{7}$$

for some fixed positive real number M. The left side of the inequality is a quadratic function of r, which we write as $a_\sigma r^2 + 2b_\sigma r + c_\sigma$ with positive a_σ. Then, we solve the next maximization problem:

$$\max_{\sigma, r \in \mathcal{I}^{(1)}} \quad \sum_{i=1}^{n} \pi_{\sigma(i)}^{(i)}(r)$$

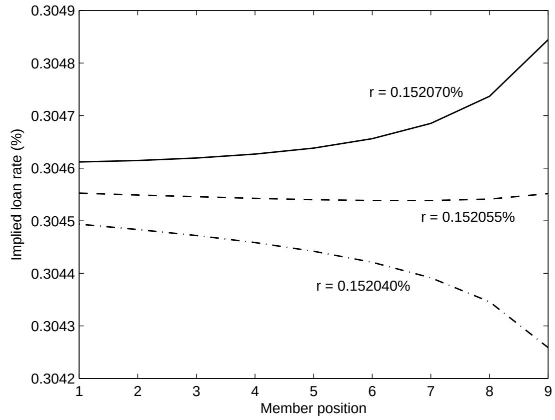

Figure 14: Implied loan rates for members on positions 1 to 9: the number of members is 10, the admissible rates are 0.15204%, 0.152055%, and 0.15207%, and the deposit rate (r_d) is 0.3%.

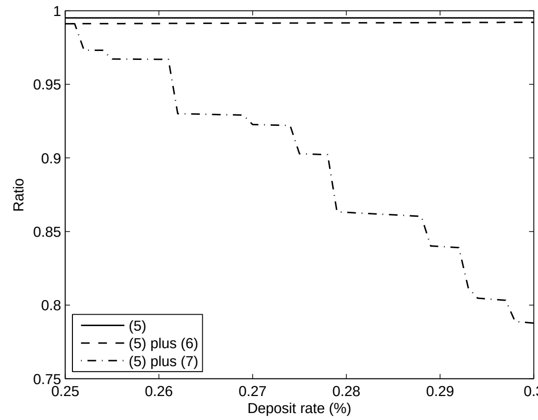

Figure 15: Ratios of maximal values of (5), (5)+(6), and (5)+(7) to that of (1): $n = 5$, $M = 0.013$, and the $r_l^{(i)}$ are 0.50%, 0.55%, 0.60%, 0.65%, and 0.70%.

$$\text{s.t.} \qquad a_\sigma r^2 + 2b_\sigma r + c_\sigma \leq M. \tag{8}$$

We note that the objective function is linear in r with a nonnegative slope as argued in the proof of Theorem 4.1. Therefore, for any fixed σ such that (8) is feasible, i.e., there exists a nonnegative r that satisfies the constraint, the best rate for that order is the largest real number in the intersection of $\mathcal{I}^{(1)}$ and the interval obtained from (8). From this reasoning, the next result easily follows.

Proposition 4.6 *The optimal rate for the problem* (5) *plus* (7) *is given by* $\max_{\sigma\in\mathcal{S}}\Big\{\min\{r_\sigma, q\} : r_\sigma \geq p,$ *and* (8) *is feasible*$\Big\}$ *where* $\mathcal{I}^{(1)} = [p, q]$, *if nonempty, and* $r_\sigma = a_\sigma^{-1}\Big(-b_\sigma + \sqrt{b_\sigma^2 - a_\sigma(c_\sigma - M)}\Big)$.

Unfortunately, the above optimization problem is formulated as an MIP with nonlinear constraints at best. There is in general no fast solution technique known for such problems. Since it is not our main concern to find an efficient algorithm for nonlinear MIPs in this work, we leave this issue as a future consideration. In the numerical example below, we conduct an exhaustive search to find the optimal order and the optimal rate.

Remark 4.7 Instead of finding r for given $r_l^{(i)}$'s and other parameters, we can consider the problem of finding loan rates which result in equal discounted extra profits. Those *implied* loan rates with a suitable order σ then make $a_\sigma r^2 + 2b_\sigma r + c_\sigma = 0$. To find such rates, we begin with the member who is in the n-th position. For that member, the discounted extra profit does not involve any loan rate. (Recall that $\mathfrak{L}_n^{(i)}(r) = 0$ in Section 2.) With all other parameters fixed, the discounted extra profit of this member is determined. Then, by equating this with other members' discounted profits, the implied loan rates can be calculated. In addition, it is straightforward to check that in order to make these rates greater than or equal to the deposit rate, which is one of our assumptions, the rates r_d and r should satisfy the following condition:

$$r\left\{\frac{n-k}{(1+r_d)^{n-k}-1}+1+\frac{1}{r_d}-k\right\}\geq 1,\quad k=1,\ldots,n-1.$$

Some examples are presented in Figure 14 which shows implied loan rates for members in a ROSCA with $n=10$. Note that there is no implied loan rate for the member in the n-th position. The figure shows that implied loan rates are affected by the rate r. As r approaches the lower bound, the implied loan rates decrease in member position, while they increase in position as r gets bigger. Hence, for a small and suitable r, it seems possible to have an order that achieves fairness and increasing discounted extra profits at the same time as long as loan rates are appropriately set. But, as r increases, such orders are reversed as implied from numerical examples, suggesting the existence of a trade-off between efficient orders and fair orders.

Before we end this subsection, we compare the solutions of optimization problems discussed so far. In Figure 15, we show the percentage values of optimal solutions of (5), (5) plus (6), and (5) plus (7) with respect to the optimal value of (1), which is solved using the commercial optimization software CPLEX via the above MIP formulation. We observe that at least in this example the first two subproblems achieve optimal values that are quite close to the optimum of the original problem while those of the subproblem with fairness consideration decrease as the deposit rate increases.

4.3 Defaultable case

For the rest of this section, we discuss the optimal design of a ROSCA incorporating default risk as done in Section 3. We assume the constant hazard rate model based on the exponential failure distribution with the hazard rate $\lambda\leq\log\left\{(1+r_l^{(1)})/(1+r_d)\right\}$. Then, similarly as in Section 3.3 and 4.1, we have the following optimization problem:

$$\max_{\sigma,r\in\bar{\mathcal{I}}^{(1)}}\sum_{i=1}^{n}\bar{\pi}_{\sigma(i)}^{(i)}(r),\tag{9}$$

where $\bar{\mathcal{I}}^{(1)}$ is the set of admissible rates for a defaultable ROSCA with homogeneous members that have deposit rate r_d and loan rate $r_l^{(1)}$, $\bar{\pi}_{\sigma(i)}^{(i)}(r):=\bar{\Pi}_k^{(i)}(r)/(1+r_d)^k$, $\bar{\Pi}_k^{(i)}(r):=\sum_{t=1}^n\Pi_{k,t}^{(i)}(r)\cdot Q_t$, and

$$\Pi_{k,t}^{(i)}(r):=\begin{cases}-\mathfrak{D}_t(1+r_d)^{k-t}, & \text{if } k>t;\\ \mathfrak{M}_k(r)-\mathfrak{D}_k-\mathfrak{L}_k^{(i)}(r), & \text{otherwise}.\end{cases}$$

We note that (9) is a defaultable version of (5). The more general case can be handled using an MIP formulation as done in Section 4.1. Hence, we rather focus on some analytical results related to (9).

Theorem 4.8 *Suppose that* (9) *is feasible. Then, the optimality for the problem is achieved at the right endpoint of* $\bar{\mathcal{I}}^{(1)}$ *with an optimal order* $\sigma(i)=n+1-i$ *for* $i=1,\ldots,n$.

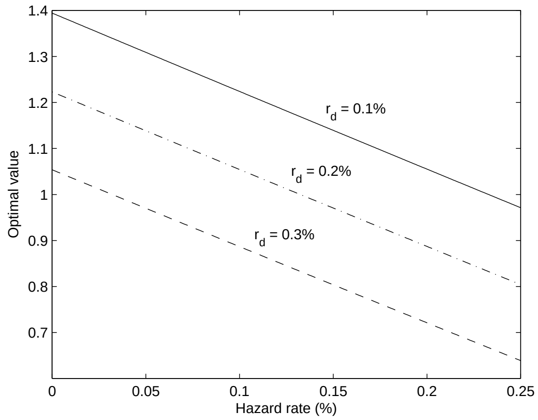

Figure 16: Optimal values of (9) as a function of λ when $n = 10$ and $r_l^{(i)}$'s are equally spaced between 0.6% and 1.05% with r_d at 0.1%, 0.2% and 0.3%.

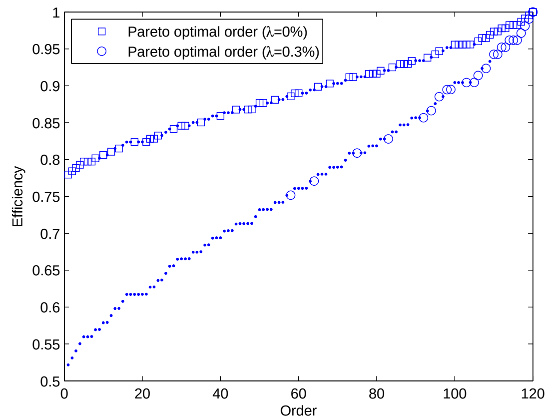

Figure 17: Efficiencies of orders when $n = 5$, $r = 0.4\%$, $r_d = 0.4\%$, and $r_l^{(i)}$'s are equally spaced between 0.8% and 1.0% with λ at 0% and 0.3%.

Intuitively, one expects that the optimal value of (9) decreases as the hazard rate λ increases, and this is numerically verified in Figure 16. This happens because the feasible region becomes smaller in λ. See Figure 10. As for Pareto optimality, a reasonable alternative measure of optimality, a result similar to Proposition 4.3 can be shown using the same line of arguments. Hence, rather than repeating the same idea, we provide Figure 17 which compares the efficiencies between the orders in the benchmark case and the defaultable case. It is easy to see that the optimal values are smaller in the defaultable case as the objective function decreases in λ in the feasible region. More interestingly, the efficiencies as well are smaller in the defaultable case although the percentage depends on a set of parameters.

5. Concluding Remarks

In this paper, we studied the optimal design of one well-known informal microfinance system, rotating savings and credit association or ROSCA. By considering banking transactions which result in the same cash inflows as a ROSCA, we defined a discounted extra profit that an investor can earn by participating in the system and we formulated the design problem as optimization problems.

When members are homogeneous, we found a complete solution to the main problem with the feasibility condition and the optimal rate. One notable finding is that, as the interest rate margin becomes not favorable, a ROSCA can contribute to the wealth of each member in the system. Also, the rate and the number of participants are affected by the banking deposit and loan rates. To further study practical implications of a ROSCA, we studied the issues of delayed investment opportunities and fairness. These essentially add constraints to the original formulation which can be regarded as partial answers to those issues. For the former, we added a condition of increasing discounted extra profits in member position as a compensation for opportunity costs. For the latter, a constraint such that the sum of squared deviation of each extra profit from the average is bounded by a fixed value is considered in order to achieve a fair distribution of the total extra profit from the system. For the defaultable case, we found a sufficient condition that makes a ROSCA still attractive to its members. Such a condition is satisfied when the interest rate margin is sufficiently large and the hazard rate is below some bound which depends on the deposit rate and the loan rate.

A similar analysis is done for the case of heterogeneous members, which is more realistic because loan rates can be different according to credit ratings of customers. Then, we showed that it is possible to formulate the original problem in the form of mixed integer programming which can be solved quite fast by any of the commercial optimization packages. We next presented a related suboptimal but still realistic problem, and found the optimal order of members and the optimal rate. The main finding is that the optimal value is achieved when members with higher loan rates are in earlier positions, that is, receiving the pot earlier. This way, the system is optimized in terms of the total extra profits, however, other practical issues such as fairness may require suboptimal orders. As in the case of homogeneous members, we considered two additional issues and found that in the case of fairness, the optimal order tends to be reversed if the rate r is sufficiently large. A partial solution to the investment opportunity costs is also suggested, which parallels the homogeneous case. For both cases, two additional constraints make a system more attractive to members, but from the viewpoint of maximizing the total extra profit, these constraints yield lower performances of a ROSCA. The analysis for the defaultable case is also carried out for heterogeneous members. It turns out that even though there is a default risk in a ROSCA, consistent results with the benchmark case are obtained as long as the hazard rate is small enough.

To sum up, we addressed some of the issues that have not been fully answered in the ROSCA literature, among which the system design together with the possible heterogeneity of members is the main theme. The resulting optimal design could help establish ROSCAs as effective funding sources for small and medium-sized businesses especially in developing economies. It has been noted in the literature that such microfinance services play an important role in building sustainable enterprises. The design maximizes their extra profits and it could lower their costs of capital so that entrepreneurs can avoid paying very high rates of interest to start businesses, sustain their cash flow management and foster their recapitalization. Additionally, we addressed the issues of increasing returns and fairness and incorporated default risks of a ROSCA by considering a hazard rate model. There are two relevant topics that can be investigated in future research. One is an empirical research of ROSCAs from the perspective of the optimal design we proposed in this paper. The other is a thorough study of bidding ROSCAs which are also quite a popular form of microfinance services observed in many countries around the globe.

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Appendix: Proofs

Proof of Lemma 3.1 Firstly, it can be easily checked that $f(1) < 0$, $f(n) > 0$, and $f'(k) > 0$ for all k because

$$f'(k) = \begin{cases} n + \frac{n(1+r_l)^k \log(1+r_l)}{r_l(1+r_l)^n}, & \text{if } r_l > 0; \\ 2n, & \text{if } r_l = 0. \end{cases}$$

This observation implies that $f(\cdot)$ is strictly increasing. Hence, $f(\cdot)$ has exactly one root k_0 in $(1, n)$.

The rest of the proof is about the increasing property of $h(\cdot)$ in $\{1, \cdots, \lceil k_0 - 1 \rceil\}$ and $\{\lfloor k_0 + 1 \rfloor, \cdots, n\}$. It is enough to check for all k in two sets above, $f(k+1)g(k) - f(k)g(k+1) \geq 0$. Let $l = 1/(1+r_l)$ and

$d = 1/(1+r_d)$. Rewriting the above inequality, we need to check for all $k = 1, \cdots, \lceil k_0 - 1 \rceil - 1$,

$$(n-k)\, l^{n-k} + n - \sum_{j=1}^{n-k-1} l^j + \left(k - \sum_{j=1}^{n-k-1} l^j \right) d^{-k} - \left(1 + l^{n-k}\right) \sum_{j=0}^{k} d^{-j} \geq 0 \tag{10}$$

We claim that for all $x \in [0,1]$ and for all $n \geq k$,

$$\psi(x,k,n) := (n-k)x^{n-k} + n + \frac{k}{x^k} - \frac{1 + x - x^n - x^{n+1}}{(1-x)x^k} \geq 0.$$

Let $\rho(x) = (1-x)x^k \psi(x,k,k)$. Then, $\rho'(x) = -k - 1 + k(k+1)x^{k-1} - (k^2-1)x^k$. Since $\rho'(1) = 0$ and $\rho'(\cdot)$ is increasing on [0,1] as easily checked by looking at ρ'', we have $\rho'(x) \leq 0$ for all $x \in [0,1]$. Therefore, $\rho(\cdot)$ is decreasing on [0,1] with $\rho(1) = 0$, and as a result, we get $\rho(x) \geq 0$ for all $x \in [0,1]$, i.e., $\psi(x,k,k) \geq 0$.

As an induction step, let us assume that $\psi(x,k,k+m) \geq 0$ for some nonnegative value m. One can check that

$$\psi(x,k,k+m+1) = \psi(x,k,k+m) + \eta(x),$$

where $\eta(x) = (m+2)x^{m+1} - (m+1)x^m + 1$. Since $\eta'(x) = (m+1)x^{m-1}\{(m+2)x - m\} = 0$, we know that η has a minimum value at $m/(m+2)$ in $[0,1]$ and $\eta(m/(m+2)) \geq 0$. This implies that $\eta(x) \geq 0$ for all $x \in [0,1]$. Hence, $\psi(x,k,k+m+1) \geq 0$ for all $x \in [0,1]$. Thus, the induction step is complete and we have proved that $\psi(x,k,n) \geq 0$ for all $x \in [0,1]$ and for all $n \geq k$.

The first step is to prove the inequality (10) on the first set. Choose k in $\{1, \cdots, \lceil k_0 - 1 \rceil - 1\}$. Recall that $f(k) < f(k+1) < 0$ for all $k = 1, \cdots, \lceil k_0 - 1 \rceil - 1$, and $f(k+1) < 0$ implies $\sum_{j=1}^{n-k-1} l^j > k$. Then,

$$\begin{aligned}\text{Left hand side of (10)} &\geq (n-k)\, l^{n-k} + n - \sum_{j=1}^{n-k-1} l^j + \left(k - \sum_{j=1}^{n-k-1} l^j \right) l^{-k} - \left(1 + l^{n-k}\right) \sum_{j=0}^{k} l^{-j} \\ &= \psi(l,k,n)\end{aligned}$$

Since $l \in [0,1]$, the inequality (10) holds.

The next step is to prove that $h(k) \leq h(k+1)$ for all $k = \lfloor k_0 + 1 \rfloor, \cdots, n-1$. Recall that $f(k+1) > f(k) > 0$ for all $k = \lfloor k_0 + 1 \rfloor, \cdots, n-1$. Rewriting the inequality, it is enough to show that for all $k = \lfloor k_0 + 1 \rfloor, \cdots, n-1$,

$$\left(n - k - \sum_{j=0}^{k-1} d^{-j} \right) l^{n-k} + n - \sum_{j=1}^{n-k-1} l^j - \sum_{j=0}^{k-1} d^{-j} + \left(k - 1 - \sum_{j=1}^{n-k} l^j \right) d^{-k} \geq 0. \tag{11}$$

Since $k - 1 - \sum_{j=1}^{n-k} l^j > 0$ and $k - \sum_{j=1}^{n-k-1} l^j > 0$, if $n - k - \sum_{j=0}^{k-1} d^{-j} \geq 0$, there is nothing to prove.

Now, we assume that $n-k-\sum_{j=0}^{k-1} d^{-j} < 0$. Then,

$$\begin{aligned}\text{Left hand side of (11)} &\geq \left(n-k-\sum_{j=0}^{k-1} d^{-j}\right) d^{n-k} + n - \sum_{j=1}^{n-k-1} d^j - \sum_{j=0}^{k-1} d^{-j} + \left(k-1-\sum_{j=1}^{n-k} d^j\right) d^{-k} \\ &= \psi(d,k,n)\end{aligned}$$

Since $d \in [0,1]$, the proof is now done. ■

Proof of Theorem 3.2 First, assume that r is admissible. Then, by definition, $f(k)r+g(k) \geq 0$ for $k = 1,\cdots,n$. Since $f(\cdot)$ is an increasing function and k_0 is the only root of $f(\cdot)$, $f(k) < 0$ for $k = 1,\cdots,\lceil k_0 - 1\rceil$ and $f(k) > 0$ for $k = \lfloor k_0+1\rfloor,\cdots,n$. Therefore, we easily see that, for $k_1 \geq \lfloor k_0+1\rfloor$ and $k_2 \leq \lceil k_0-1\rceil$, we have

$$h(k_1) \leq r \leq h(k_2).$$

Hence, $\mathcal{I}$ is nonempty and r is in the interval.

Now, we prove the converse, i.e., any element in $\mathcal{I}$ is admissible. Choose any $r \in \mathcal{I}$. This leads to $f(k)r+g(k) \geq 0$ for all $k \in \{1,\ldots,\lceil k_0-1\rceil, \lfloor k_0+1\rfloor,\ldots,n\}$. If k_0 is not an integer, then we obtain the desired result because $\Pi_k(r) \geq 0$ holds for every $k = 1,\ldots,n$. If k_0 is an integer, then $f(k_0) = 0$. But, if the inequality $g(k_0) \geq 0$ is true, then we still have $\Pi_k(r) \geq 0$ for all $k = 1,\ldots,n$. We claim that indeed $g(k_0)$ is nonnegative when k_0 is an integer.

In the proof of Lemma 3.1, we show that

$$h(k) \leq h(k+1)$$

for $k = 1,\ldots,\lceil k_0-1\rceil-1$. The inequality remains true as long as $f(k+1) < 0$ even when k is not an integer. (The proof above still applies to non-integer k's.) Of course, here, we consider $f(\cdot)$ as a function defined on $\mathbb{R}_+$ by completing summations. Then, for a non-integer k, we get $h(a) \leq h(a+1) \leq \cdots \leq h(k) \leq h(k+1)$ where $a = k - \lfloor k\rfloor \in (0,1)$. It is easy to check that $f(1) < 0$ and $g(1) = n-1-\sum_{j=1}^{n-1}(1+r_l)^{-j} \geq 0$. The same inequalities still hold for any number in $(0,1)$. This makes $h(a) \geq 0$ and thus we get $g(k+1) \geq 0$ because it is assumed that $f(k+1) < 0$. Consequently by the continuity of $g(\cdot)$, we have $g(k_0) \geq 0$. Therefore, r becomes an admissible rate regardless of whether k_0 is an integer or not.

For the last statement, we see that the objective function in (1) is simply $\left(\sum_{k=1}^n f(k)(1+r_d)^{-k}\right)r + \sum_{k=1}^n g(k)(1+r_d)^{-k}$. Each $f(k)$ is increasing in r_l. Then, we observe that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\frac{f(k)}{(1+r_d)^k} &= \sum_{k=1}^{n}\left\{\frac{k-1}{(1+r_d)^k} - \sum_{j=1}^{n-k}\left(\frac{1}{1+r_l}\right)^j\left(\frac{1}{1+r_d}\right)^k\right\} \\ &\geq \sum_{k=1}^{n}\left\{\frac{k-1}{(1+r_d)^k} - \sum_{j=1}^{n-k}\left(\frac{1}{1+r_d}\right)^{j+k}\right\}\end{aligned}$$

$$= \sum_{k=1}^{n} \frac{k-1}{(1+r_d)^k} - \sum_{k=1}^{n} \sum_{i=k+1}^{n} \left(\frac{1}{1+r_d}\right)^i = 0$$

where we used $r_l \geq r_d$. The double summation on the last line is equal to the first term on that line after we interchange the order of summations. Hence, the objective function (1) is a linear function with nonnegative slope, achieving its maximum at the right endpoint $-g(1)/f(1) = h(1)$. ■

Proof of Corollary 3.3 Let $\widetilde{r} := r_d = r_l$. If $\widetilde{r} = 0$, then $g(k) = 0$ for all k and thus $\Pi_k(r) = f(k)r$. Also, $h(1) = h(n) = 0$, which results in $\mathcal{I} = \{\widetilde{r}/2\} = \{0\}$. If $n = 2$, then it is simple to check that $h(1) = h(2) = \widetilde{r}/2$, so $\mathcal{I} = \{\widetilde{r}/2\}$.

Assume that $\widetilde{r} > 0$ and $n > 2$. Since $(1+\widetilde{r})^n \geq R := 1 + \widetilde{r}n + \widetilde{r}^2 n(n-1)/2 + \widetilde{r}^3 n(n-1)(n-2)/6$,

$$\begin{aligned}
\frac{g(1)}{f(1)} - \frac{g(n)}{f(n)} &= \frac{1}{n}\left\{\frac{(1+\widetilde{r})^n - 1 - \widetilde{r}}{\widetilde{r}(n-1)} - \frac{\widetilde{r}(n-1)}{1-(1+\widetilde{r})\,(1+\widetilde{r})^{-n}}\right\} \\
&\geq \frac{1}{n}\left\{\frac{R-1-\widetilde{r}}{\widetilde{r}(n-1)} - \frac{\widetilde{r}(n-1)}{1-(1+\widetilde{r})R^{-1}}\right\} \\
&= \frac{\widetilde{r}^2(n-2)}{36 + 18\widetilde{r}n + 6\widetilde{r}^2 n(n-2)}\left\{3 + 6\widetilde{r} + \widetilde{r}^2 n(n-2)\right\} > 0.
\end{aligned}$$

This means that $-g(n)/f(n) > -g(1)/f(1)$, which implies that $\mathcal{I}$ is empty. ■

Proof of Proposition 3.5 We first note that

$$\begin{aligned}
g(k+1) - (1+r_d)g(k) &= -(1+r_d)^k + (1+r_l)^{-(n-k)} - r_d\, g(k) \\
&= -(1+r_d)^k + (1+r_l)^{-(n-k)} - r_d n + (1+r_d)^k - 1 + r_d \sum_{j=1}^{n-k} (1+r_l)^{-j} \\
&= (1+r_l)^{-(n-k)} - 1 - r_d\left(n - \sum_{j=1}^{n-k}(1+r_l)^{-j}\right).
\end{aligned}$$

This expression becomes $-r_d k$ when $r_l = 0$. Thus, $\Delta_g(k)$ is strictly negative. Similarly for $r_l > 0$, we have strictly negative $\Delta_g(k)$'s for all k. Consequently, a nonnegative rate r satisfies (3) if and only if $\Delta_f(k) > 0$ for all k.

Now, it remains to check $f(k+1) > (1+r_d)f(k)$ for all k. If $r_l = 0$, then $r_d = 0$ and thus the inequality to check becomes $f(k+1) > f(k)$. Since $f(\cdot)$ is strictly increasing with $f'(\cdot) = 2n$, this inequality is trivial. Let us assume $r_l > 0$. In a straightforward manner, we have

$$\begin{aligned}
& f(k+1) - (1+r_d)f(k) > 0 \\
\Leftrightarrow\quad & k - \sum_{j=1}^{n-k-1}\left(\frac{1}{1+r_l}\right)^j > (1+r_d)\left\{k - 1 - \sum_{j=1}^{n-k}\left(\frac{1}{1+r_l}\right)^j\right\} \\
\Leftrightarrow\quad & \frac{k+1}{1+r_d} - \sum_{j=0}^{n-k-1}\left(\frac{1}{1+r_l}\right)^j \frac{1}{1+r_d} > k - 1 - \sum_{j=0}^{n-k-1}\left(\frac{1}{1+r_l}\right)^j \frac{1}{1+r_l}
\end{aligned}$$

$$\Leftrightarrow \quad (1+r_l)\Big(k+1-(k-1)(1+r_d)\Big) > (r_l - r_d)\sum_{j=0}^{n-k-1}\left(\frac{1}{1+r_l}\right)^j.$$

The remaining steps are obvious and thus omitted.

The second statement is straightforward to prove, using the arguments in the proof of Theorem 3.2. Hence, we leave the details to the reader. ■

Proof of Lemma 3.8 Since $\bar{f}(1) < 0$, $\bar{f}(n) > 0$, and $\bar{f}(\cdot)$ is continuous, there exists at least one root in the interval $(1, n)$. Assume by contradiction that there are two or more roots in the interval $(1, n)$. Then, there exists c in $(1, n)$ such that $\bar{f}(c) = 0$ and $\bar{f}'(c) \le 0$. However, if $\bar{f}(c) = 0$ for some c, then $\bar{f}'(c) > 0$ because $\bar{f}'(k) = -\lambda \bar{f}(k) + ne^{-\lambda(k-1)}\left\{1 + (1+r_l)^{k-n}\log(1+r_l)/r_l\right\}$ if $r_l > 0$; $\bar{f}'(k) = -\lambda\bar{f}(k) + 2ne^{-\lambda(k-1)}$ if $r_l = 0$. Hence, $\bar{f}(\cdot)$ has exactly one root k_0 in $(1, n)$. The rest of the proof is the same with the proof of Lemma 3.1 by replacing d with $\left\{e^{\lambda}(1+r_d)\right\}^{-1}$. ■

Proof of Theorem 4.1 The first part of the proof computes the optimal rate and the second part finds an optimal order. For notational convenience, we write $\Pi_k^{(i)}(r)$ as $f^{(i)}(k)r + g^{(i)}(k)$. Note that the objective function of (5) is

$$\left(\sum_{i=1}^{n}\frac{f^{(i)}(\sigma(i))}{(1+r_d)^{\sigma(i)}}\right) r + \sum_{i=1}^{n}\frac{g^{(i)}(\sigma(i))}{(1+r_d)^{\sigma(i)}}.$$

Then, as in the proof of Theorem 3.2, we use $r_d \le r_l^{(i)}$ in the following computations:

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\frac{f^{(\sigma^{-1}(k))}(k)}{(1+r_d)^k} &= \sum_{k=1}^{n}\left\{\frac{k-1}{(1+r_d)^k} - \sum_{j=1}^{n-k}\left(\frac{1}{1+r_l^{(\sigma^{-1}(k))}}\right)^j\left(\frac{1}{1+r_d}\right)^k\right\} \\
&\ge \sum_{k=1}^{n}\left\{\frac{k-1}{(1+r_d)^k} - \sum_{j=1}^{n-k}\left(\frac{1}{1+r_d}\right)^{j+k}\right\} \\
&= \sum_{k=1}^{n}\frac{k-1}{(1+r_d)^k} - \sum_{k=1}^{n}\sum_{i=k+1}^{n}\left(\frac{1}{1+r_d}\right)^i \\
&= \sum_{k=1}^{n}\frac{k-1}{(1+r_d)^k} - \sum_{i=2}^{n}\frac{i-1}{(1+r_d)^i} = 0.
\end{aligned}$$

Hence, the objective function has a nonnegative slope, thus it is maximized at the right end of $\mathcal{I}^{(1)}$.

As for an optimal order, we first fix n and r, and let $\alpha = (1+r_d)^n/(1+nr)$. For the rest of this proof, we write s_k for $\sigma^{-1}(k)$ for notational simplicity. (And s'_k for $(\sigma')^{-1}(k)$, and so on.) Our objective is to show that $\sigma^*(i) = n+1-i$, or equivalently, $s^*_k = n+1-k$ is optimal. Recall that

$$\sum_{i=1}^{n}\pi_{\sigma(i)}^{(i)}(r) = \sum_{k=1}^{n}\pi_k^{(s_k)}(r) = \sum_{k=1}^{n}\frac{\mathfrak{M}_k(r) - \mathfrak{D}_k}{(1+r_d)^k} - \sum_{k=1}^{n}\frac{\mathfrak{L}_k^{(s_k)}(r)}{(1+r_d)^k}.$$

Since the first summation on the right hand is independent of $r_l^{(i)}$'s, it is enough to show that the second summation is minimized when $s_k = n+1-k$. In other words, we prove that for any permutation s,

$$\begin{aligned}
\alpha \sum_{k=1}^{n} \frac{\mathfrak{L}_k^{(n+1-k)}(r)}{(1+r_d)^k} &= \alpha \sum_{i=1}^{n} \frac{\mathfrak{L}_{n+1-i}^{(i)}(r)}{(1+r_d)^{n+1-i}} \\
&= \sum_{i=1}^{n} \frac{(1+r_d)^{i-1}}{r_l^{(i)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(i)}} \right)^{i-1} \right\} \\
&\leq \sum_{k=1}^{n} \frac{(1+r_d)^{n-k}}{r_l^{(s_k)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s_k)}} \right)^{n-k} \right\} = \alpha \sum_{k=1}^{n} \frac{\mathfrak{L}_k^{(s_k)}(r)}{(1+r_d)^k}
\end{aligned}$$

For a proof, we use an induction. Suppose $n = 2$. Then, there are only two possible orders, and we observe that

$$\alpha \sum_{k=1}^{2} \frac{\mathfrak{L}_k^{(3-k)}(r)}{(1+r_d)^k} = \frac{1+r_d}{1+r_l^{(2)}} \leq \frac{1+r_d}{1+r_l^{(1)}} = \alpha \sum_{k=1}^{2} \frac{\mathfrak{L}_k^{(s_k)}(r)}{(1+r_d)^k}$$

where $s_k = k$. We suppose that the above inequality holds for up to $m-1$. Let $n = m$. For any given permutation s, consider another permutation s' such that

$$s_1 = p = s'_j, \quad s_j = m = s'_1.$$

For all other k, $s_k = s'_k$. If $s_1 = m$, then $s' = s$. By construction, we note that s' is a bijection from $\{2, \ldots, m\}$ to $\{1, \ldots, m-1\}$. We then proceed as follows. Since the statement holds for $n = m-1$,

$$\begin{aligned}
&\sum_{i=1}^{m} \frac{(1+r_d)^{i-1}}{r_l^{(i)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(i)}} \right)^{i-1} \right\} \\
&= \sum_{i=1}^{m-1} \frac{(1+r_d)^{i-1}}{r_l^{(i)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(i)}} \right)^{i-1} \right\} + \frac{(1+r_d)^{m-1}}{r_l^{(m)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(m)}} \right)^{m-1} \right\} \\
&\leq \sum_{k=2}^{m} \frac{(1+r_d)^{m-k}}{r_l^{(s'_k)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s'_k)}} \right)^{m-k} \right\} + \frac{(1+r_d)^{m-1}}{r_l^{(m)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(m)}} \right)^{m-1} \right\} \\
&= \sum_{k=1}^{m} \frac{(1+r_d)^{m-k}}{r_l^{(s'_k)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s'_k)}} \right)^{m-k} \right\}.
\end{aligned}$$

The above reasoning tells us that if $s' = s$, then we are done, and if $s' \neq s$, then it is enough to show the next inequality:

$$\begin{aligned}
&\frac{(1+r_d)^{m-1}}{r_l^{(s'_1)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s'_1)}} \right)^{m-1} \right\} + \frac{(1+r_d)^{m-j}}{r_l^{(s'_j)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s'_j)}} \right)^{m-j} \right\} \\
&\leq \frac{(1+r_d)^{m-1}}{r_l^{(s_1)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s_1)}} \right)^{m-1} \right\} + \frac{(1+r_d)^{m-j}}{r_l^{(s_j)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s_j)}} \right)^{m-j} \right\}.
\end{aligned}$$

Using the relationship $s_1 = p = s'_j$ and $s_j = m = s'_1$, this is simply

$$\begin{aligned}
&\frac{(1+r_d)^{m-1}}{r_l^{(m)}}\left\{1-\left(\frac{1}{1+r_l^{(m)}}\right)^{m-1}\right\}+\frac{(1+r_d)^{m-j}}{r_l^{(p)}}\left\{1-\left(\frac{1}{1+r_l^{(p)}}\right)^{m-j}\right\}\\
&\leq \frac{(1+r_d)^{m-1}}{r_l^{(p)}}\left\{1-\left(\frac{1}{1+r_l^{(p)}}\right)^{m-1}\right\}+\frac{(1+r_d)^{m-j}}{r_l^{(m)}}\left\{1-\left(\frac{1}{1+r_l^{(m)}}\right)^{m-j}\right\}.
\end{aligned}$$

But, then this is nothing but to show

$$\begin{aligned}
&(1+r_d)^{m-j}\left\{\sum_{i=1}^{m-j}\left(\frac{1}{1+r_l^{(p)}}\right)^{i}-\sum_{i=1}^{m-j}\left(\frac{1}{1+r_l^{(m)}}\right)^{i}\right\}\\
&\leq (1+r_d)^{m-1}\left\{\sum_{i=1}^{m-1}\left(\frac{1}{1+r_l^{(p)}}\right)^{i}-\sum_{i=1}^{m-1}\left(\frac{1}{1+r_l^{(m)}}\right)^{i}\right\},
\end{aligned}$$

which is obvious from the condition $0 \leq r_d \leq r_l^{(p)} \leq r_l^{(m)}$. The proof is complete. ■

Proof of Proposition 4.3 Let us set

$$\Delta_k^{(i)}(r) = \pi_{k+1}^{(i)}(r) - \pi_k^{(i)}(r) = C + \frac{r_d}{(1+r_d)^{k+1}}\sum_{j=1}^{n-k}\frac{1+nr}{(1+r_l^{(i)})^j} + \frac{1+nr}{(1+r_d)^{k+1}(1+r_l^{(i)})^{n-k}},$$

where

$$C = \frac{\mathfrak{M}_{k+1}(r)-\mathfrak{D}_{k+1}}{(1+r_d)^{k+1}} - \frac{\mathfrak{M}_k(r)-\mathfrak{D}_k}{(1+r_d)^k}.$$

Then, $\Delta_k^{(i)}$ decreases in i because $r_l^{(1)} \leq r_l^{(2)} \leq \cdots \leq r_l^{(n)}$. Therefore, $\min_k \Delta_k^{(n)}(r) > 0$ is enough to guarantee that $\min_k \Delta_k^{(i)}(r) > 0$ for all $i = 1, \ldots, n-1$. Similarly, $\max_k \Delta_k^{(1)}(r) < 0$ implies that $\max_k \Delta_k^{(i)}(r) < 0$ for all i's.

Suppose that $\min_k \Delta_k^{(n)}(r) > 0$, i.e., $\pi_k^{(n)}(r)$ is strictly increasing in k. When the current order is changed, there should be at least one member, say i, who is moved from a later position to an earlier position. Since $\pi_k^{(i)}(r)$ is also strictly increasing in k, this member is worse off by a new order. Therefore, every order is Pareto optimal. In a similar fashion, when $\max_k \Delta_k^{(1)}(r) < 0$ and the current order is changed, then there exists a member whose profit is reduced. Hence, every order is Pareto optimal in this case as well. ■

Proof of Theorem 4.8 The first part is similar to the proof of Theorem 4.1, utilizing the inequality $e^\lambda(1+r_d) \leq 1 + r_l$. For the second part, we first fix n and r, let $\beta = \{e^\lambda(1+r_d)\}^n/(1+nr)$, and write s_k for $\sigma^{-1}(k)$ for notational simplicity. (And s'_k for $(\sigma')^{-1}(k)$, and so on.) Our objective is to show that $\sigma^*(i) = n+1-i$, or equivalently, $s^*_k = n+1-k$ is optimal. Recall that

$$\sum_{i=1}^{n}\bar{\pi}_{\sigma(i)}^{(i)}(r) = \sum_{k=1}^{n}\bar{\pi}_k^{(s_k)}(r) = \sum_{k=1}^{n}\left[\frac{(\mathfrak{M}_k(r)-\mathfrak{D}_k)e^\lambda}{\{e^\lambda(1+r_d)\}^k} - \sum_{t=1}^{k-1}\frac{\mathfrak{D}_t Q_t}{(1+r_d)^t}\right] - e^\lambda\sum_{k=1}^{n}\frac{\mathfrak{L}_k^{(s_k)}(r)}{\{e^\lambda(1+r_d)\}^k}.$$

Since the first summation on the right hand is independent of $r_l^{(i)}$'s, it is enough to show that the second summation is minimized when $s_k = n+1-k$. Fix any permutation s. Then, we get

$$\begin{aligned}
\beta \sum_{k=1}^{n} \frac{\mathfrak{L}_k^{(n+1-k)}(r)}{\{e^{\lambda}(1+r_d)\}^k} &= \beta \sum_{i=1}^{n} \frac{\mathfrak{L}_{n+1-i}^{(i)}(r)}{\{e^{\lambda}(1+r_d)\}^{n+1-i}} \\
&= \sum_{i=1}^{n} \frac{\{e^{\lambda}(1+r_d)\}^{i-1}}{r_l^{(i)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(i)}} \right)^{i-1} \right\} \\
&\leq \sum_{k=1}^{n} \frac{\{e^{\lambda}(1+r_d)\}^{n-k}}{r_l^{(s_k)}} \left\{ 1 - \left(\frac{1}{1+r_l^{(s_k)}} \right)^{n-k} \right\} = \beta \sum_{k=1}^{n} \frac{\mathfrak{L}_k^{(s_k)}(r)}{\{e^{\lambda}(1+r_d)\}^k}
\end{aligned}$$

For the proof of the inequality, we adopt the method used in the proof of Theorem 4.1 with $e^{\lambda}(1+r_d)$ instead of $1+r_d$. Then, the result follows. ■