Dynamic Pricing with “BOGO” Promotion in Revenue Management

Kyoung-Kuk Kim∗, Sunggyun Park†
Korea Advanced Institute of Science and Technology
Chi-Guhn Lee‡
University of Toronto

Mar 2016

Abstract

We consider a dynamic pricing problem when a seller, facing uncertain demands, sells a single product in a finite horizon. The seller actively adopts dynamic pricing and quantity discount schemes. The proposed model is based on the assumption that each customer has random reservation prices and the purchase size depends on the posted price and discount. We particularly focus on the widely adopted promotional schemes “buy one get one free” and “50% off” and study the optimal strategic choices of the seller. Analytical results together with numerical experiments are presented to help us obtain managerial insights. Additional numerical results for a generalized model are provided so as to examine the effectiveness of promotional schemes.

KEYWORDS: Revenue Management; Price Promotion; Reservation Price; Copula

1 Introduction

Pricing is an important factor that determines retailers’ profitability. Among many success stories, the airline industry is regarded as one prominent example in which pricing optimization techniques have successfully resulted in increased revenues (Phillips, 2005). Other examples include electricity pricing, hotels and rental cars, etc. According to Sullivan (2005), companies that employ price optimization techniques were able to raise their gross margins ranging from one percent to three percent, and in some cases up to ten percent. Elmaghraby and Keskinocak (2003) point out the availability of customer data along with decision support tools for analyzing such data as well as new technologies that make price changes easy as the drivers for the development of dynamic pricing strategies. Firms in e-commerce, in particular, actively adopt dynamic pricing and experience real growth. For example, E-Bay used a dynamic pricing strategy that sold more than 20 billion dollars worth of goods in 2005 (Sahay, 2007) and Amazon made 3 million daily price changes through the month of November 2013, outperforming other retailers (Berthiaume, 2014).

∗Industrial and Systems Engineering, E-mail: kkim3128@kaist.ac.kr
†Corresponding author, Industrial and Systems Engineering, E-mail: sunggyun@kaist.ac.kr
‡Mechanical and Industrial Engineering, E-mail: cglee@mie.utoronto.ca
And yet there are many popular strategies employed by retailers other than dynamic pricing. One such example is the phrases such as “buy one get one free” (“BOGOF” for short) that we easily encounter in everyday life. This particularly assumes that demands occur in multi-units or batches and retailers need to consider optimal menus (combination of batch sizes, prices, and discounts) to customers, considering customers’ willingness-to-pay. That certainly raises the problem complexity considerably. Sometimes it is referred to as dynamic nonlinear pricing because different unit prices can be set depending on various quantities (Levin and Nediak, 2014), or simply quantity discounts because often more discounts are offered for larger purchases. Although such approaches have been dealt in the literature, it is still limited in terms of dynamic consideration of pricing and other promotional strategies together. The present work was motivated by this research gap.

Particularly interesting and prevalent promotional strategies are “BOGOF” or “50% off.” Even if it is certainly the latter that is more attractive to customers if both are offered at the same unit price, it is not so clear which is the best for a seller as there is a possibility of raising a higher revenue by selling two units through the former. Customer responses to those schemes, therefore, are the most important factor in determining effective promotional strategies and have been the target of active academic studies. For example, Sinha and Smith (2000) compare “BOGOF” and “50% off” empirically to understand the difference in customers’ perceived transaction values under different schemes. By extending this work, Li et al. (2007) divide dairy foods into four types according to stock-up characteristics and consumption levels and run a survey about customer preferences. To mention other examples, Jayaraman et al. (2013) investigate consumer’s satisfaction and repurchase intention from “BOGOF” in Malaysia. Salvi (2013) studies the effectiveness of “BOGOF” compared to other strategies in Indian apparel retail industry. In this paper, we aim to understand these popular strategies via quantitative models, which are scarce in the revenue management literature, and derive insights to support the seller’s decision making.

Our contributions can be further specified as follows. First, we develop a dynamic programming formulation in order to find the seller’s optimal choices at each sales epoch which depend on the profitability of “BOGOF” and “50% off.” It is dynamic pricing because the seller can dynamically adjust the price, say between $p$ and $p/2$, but there is an additional option to provide “BOGOF” without altering the posted price. Second, the common reservation price approach is extended to handle the case where there are two random willingness-to-pay, say $R_1$ for one unit and $R_2$ for two units. Based on the analysis of the proposed model, some managerial insights can be obtained. Lastly, we compare the proposed approach with existing dynamic (nonlinear) pricing strategies in an extended model setting. To sum up, our model formulates the seller’s problem as a stochastic dynamic program and offers a theoretical background to the above mentioned empirical results. Thomas and Chrystal (2013) give another theoretical contribution by explaining schemes such as “BOGOF” and other quantity discount methods through their relative utility pricing model, however our model considers this problem in a dynamic environment.

The rest of this paper is organized as follows. Relevant literatures are reviewed in the next section. Section 3 describes the setting and important modeling features. In Section 4, we analyze the optimal strategic choices of the seller by comparing promotional schemes “BOGOF” and “50% off” with “no promotion” case. We further conduct numerical experiments in Section 5 for a more complete understanding. In the following section, we also run several numerical experiments for a generalized model for dynamic pricing and dynamic discounting, and examine the effectiveness of promotional schemes. Section 7 concludes.
2 Literature Review

Since the study of dynamic pricing for a single product and a single demand class (Gallego and van Ryzin, 1994), the literature has flourished in many different modeling frameworks. One obvious extension is the case of multiproduct with multiple customer classes as done in Gallego and van Ryzin (1997). Another work in a multiproduct setting that is somewhat relevant is Maglaras and Meissner (2006) where multiple products request one unit of a single common resource. A relatively recent development is for the case where multiple customer classes exist for a single product. In Ding et al. (2006), the authors investigate dynamic pricing for multiple customer classes where each customer requires one unit of the product. However, there have been only a limited number of studies on multi-unit demands. Examples include Elmaghraby et al. (2008) and Wang et al. (2013). In the former, the authors look at optimal markdown designs with preannounced prices, assuming that all participants have complete information on customer demands. On the other hand, the latter formulates a stochastic dynamic programming and study optimal prices for a monopolist who sells multiple identical, imperishable items over an infinite horizon.

There are a couple of the most recent and most relevant works which we review in more details. One such paper is Levin and Nediak (2014) and they analyze optimal dynamic nonlinear pricing, including discounts and premiums. More specifically, the authors investigate the pricing policy of a monopolistic firm where customers can purchase multiple units based on their willingness-to-pay. Their model assumes several customer pools and attaches a pre-determined number of items to purchasing customers who are from the same pool. This means that customers do not make a choice between batches of different size, which is indeed pointed out in Levin and Nediak (2014). The proposed model in the current paper instead assumes that every customer decides the number of items (up to two) to purchase after observing the posted price and discount.

The other related paper is Lu et al. (2014) where the authors deal with a similar problem but they focus on a dual-pricing strategy which offers a unit selling price and a quantity discount on a batch of fixed size. While presenting a joint analysis on quantity-based price differentiation and inventory control, a market demand is determined by a single random variable and the total market share is a deterministic function of posted prices. An additional random variable is then introduced to derive an imperfect correlation between the demands for unit-sales and for quantity-sales. In this paper, we instead consider that the demands for different batch sizes are determined by their own randomness. We later show that those random variables have some natural constraints. Furthermore, the dependence structure between reservation prices is quite general, modeled via copula functions. As one consequence, the approach adopted in this paper is extendable to other promotional strategies which concern multiple products such as bundling.

Since one can consider “BOGOF” as bundling two units of the target product, there is a certain similarity between such promotional schemes and product bundling, which is also a type of nonlinear pricing for multiple products. In previous works, the focus has been on the profitability of bundling where customer behaviors are often modeled via reservation prices. As early as in 60’s, Stigler (1963) shows that bundling can be profitable when reservation prices are negatively correlated. Using the same setting, Adams and Yellen (1976) demonstrate the profitability of mixed bundling, a strategy that allows both single selling and bundling. Additional insights are obtained in Schmalensee (1984) who assumes that reservation prices follow a bivariate normal distribution in the Adams-Yellen framework. This normality assumption is weakened in Long (1984). More recently, a general dependence structure for willingness-to-pay via copula functions...
has been proposed in, for example, Chen and Riordan (2013) whose contribution lies in identifying suitable conditions for the profitability of bundling under different dependence structures of reservation prices. Later in the paper, we remark on a necessary and sufficient condition for the profitability of discounting in our proposed model. Also there are many discussions on bundling from the operations management point of view. Since they are not essential in this paper, we simply refer the interested reader to Banciu et al. (2010), Bitran and Ferrer (2007), or Hitt and Chen (2005) and references therein for further information.

3 The Model

3.1 Problem formulation

Let us consider a seller who offers a single product to customers over a finite horizon. We assume that the inventory is fixed and possibly perishable. To maximize the revenue from the existing inventory, the seller chooses to offer discounts for anyone that purchases an additional item. Among many possibilities, we focus on specific promotional strategies in this paper, namely “no promotion” (n), “BOGOF” (b), and “50% off” (f). Possible prices are then p or p/2. Although this consideration seems restrictive, they are widely observed in real practices and our choice allows us to derive some analytical properties of optimal promotional strategies and offer insights.

Nevertheless, a general setting can be given as follows. This generalized version is implemented and tested in a later section for numerical experiments. Discount decisions are fully dynamic and they are combined with dynamic pricing as well. Specifically, the action space for price p and discount q is given by

\[ A \subset P \times Q, \quad P = \{ p_1, \ldots, p_k \} \subset \mathbb{R}_+, \quad Q = \{ q_1, \ldots, q_m \} \subset [0, 1] \]

for some integers k, m. Selecting \((p, q) \in A\) means that the first item is offered at the price of \(p\) and the second at \(pq\). Hence, for example, if \(q = 1\), then there is no discount; if \(q = 0\), then the additional item is free or in other words, it is effectively the strategy “BOGOF.” Therefore, the three strategies n, b, f for fixed unit price p can also be written as \((p, 1)\), \((p, 0)\), \((p/2, 1)\), respectively.

Variable \(t\) is time to maturity and it decreases from \(t = T\) to \(t = 0\), the end of the sales season. The inventory level is denoted by \(s\) in \(\{0, \ldots, S\}\) where \(S\) is the initial inventory. Customers are assumed to arrive with probability \(\lambda\) at each time step. Note that this discrete time model can be understood as an approximation to a continuous time model where customers arrivals follow a Poisson process with rate \(\tilde{\lambda}\) by setting \(\lambda = \tilde{\lambda}\Delta t\) within a time interval of size \(\Delta t\). Likewise, our approach can be easily extended to nonhomogeneous Poisson arrivals. We also assume that customers are homogeneous in the sense that their reservation prices are independent and identically distributed.

The most important feature in the model is how customer purchase probabilities are determined. Reservation price approach has been quite popular in the literature. In particular, some recent papers address customer purchasing behaviors of bundled products by correlated reservation prices (Chen and Riordan, 2013). However, there is a scarce source of information when it comes to the multi-item and single product case. One approach is to model multi-class customers who quote pre-determined batch sizes with independent Poisson arrival processes as in Levin and Nediak (2014). We do not assume separate customer pools for different batch sizes, but restrict the purchase amount to two for tractability.

4
Throughout the paper, we denote the purchase probabilities of each customer at time $t$ by $\pi_{i,t}$ where $i$ is the number of items to purchase and $i = 0, 1, 2$. Wherever we need to specify a particular menu, we use $\pi_{i,t}^a$ for $a \in \mathcal{A}$. When $\pi_{i,t}^a$ is independent of $t$, we simply write $\pi_{i,t}^a$. The main idea in computing purchase probabilities is that the differences between reservation prices and offered prices determine customer choice. In more detail, $\pi_{i,t}$’s are given as follows:

\[
\begin{align*}
\pi_{1,t} &= P \left( R_{1,t} > p, R_{1,t} - p > R_{2,t} - p(1 + q) \right), \\
\pi_{2,t} &= P \left( R_{2,t} > p(1 + q), R_{2,t} - p(1 + q) > R_{1,t} - p \right)
\end{align*}
\]

where $R_{1,t}, R_{2,t}$ stand for the reservation prices of homogeneous customers for a single item and two items, respectively.

A little thought reveals that modeling $R_{1,t}, R_{2,t}$ as correlated random variables is not enough. This is because they have the intuitive constraint: $R_{1,t} \leq R_{2,t} \leq 2R_{1,t}$. For this reason, we seek for an alternative but equivalent expression that is better suited for our purpose, that is,

\[
\begin{align*}
R_{1,t} &= \gamma^{T-1}(\alpha X + \beta Y), \\
R_{2,t} &= \gamma^{T-1}(\alpha X + 2\beta Y).
\end{align*}
\]

Consider a linear transformation $x \mapsto \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x$, and this maps the region $\{(x_1, x_2) | 0 \leq x_1 \leq x_2 \leq 2x_2 \}$ to the first quadrant. Therefore, if we define $(X', Y') = (2R_{1,t} - R_{2,t}, R_{2,t} - R_{1,t})$, then $(X', Y')$ is a bivariate random vector with the first quadrant as its state space. Hence, any bivariate model for reservation prices can be expressed as above by setting $(X', Y') = \gamma^{T-1}(\alpha X, \beta Y)$. We consider $(X, Y)$ as the basic product features that induce purchase probabilities with “factor loading” parameters $\alpha$, $\beta$, and the time depreciation factor $\gamma$.

In our discrete-time model, the optimality equation is easily found to be

\[
V_t(s) = \max_{(p,q) \in \mathcal{A}} \left\{ \lambda \pi_{1,t}(p + V_{t-1}(s - 1)) + \lambda \pi_{2,t}(p + pq + V_{t-1}(s - 2)) \\
+ \lambda \pi_{0,t}V_{t-1}(s) + (1 - \lambda)V_{t-1}(s) \right\}
\]

for $s \geq 2$. When $s = 1$, the retailer can control the price but not discount, thus

\[V_t(1) = \max_{p \in \mathcal{P}} \left\{ \lambda \pi_{1,t}'(p + \lambda(1 - \pi_{1,t}'))V_{t-1}(1) + (1 - \lambda)V_{t-1}(1) \right\}\]

where $\pi_{1,t}' = P \left( R_{1,t} > p \right)$. We set $V_t(0) \equiv 0$ for any $t$. Sometimes a modified version becomes handy. If we set $\Delta_t V_t(s) = V_t(s) - V_{t-1}(s)$ and $\Delta_i V_t(s) = V_t(s) - V_t(s - i)$ for $i = 1, 2$, then the optimality equation is re-written as

\[
\Delta_t V_t(s) = \max_{(p,q) \in \mathcal{A}} \left\{ \lambda \pi_{1,t}(p - \Delta_1 V_{t-1}(s)) + \lambda \pi_{2,t}(p + pq - \Delta_2 V_{t-1}(s)) \right\}
\]

Also, when $s = 1$, we get

\[\Delta_t V_t(1) = \max_{p \in \mathcal{P}} \left\{ \lambda \pi_{1,t}'(p - \Delta_1 V_{t-1}(1)) \right\}\]

Finally, we assume that the salvage value of the leftover items is zero, that is, $V_0(s) \equiv 0$ for any $s$, meaning that the product is not costly to dispose at the end. This is a mild assumption that can be easily relaxed.
Remark 1  It is a simple matter to check that the optimal value function $V_t(s)$ is monotone in both $s$ and $t$. Intuitively, when the inventory level is at $s + 1$, any strategy that the seller can implement with inventory $s$ is applicable to a subset of $s$ units. The extra one unit is the source of additional revenue. It is similar in the case of the remaining time. One additional sales period provides a chance for a higher revenue. This also can be proved mathematically by induction.

3.2 Interpretation of model ingredients

Previously, we introduced a 2-dimensional random vector $(X, Y)$ as a convenient modeling scheme for the correlated reservation prices $(R_{1,t}, R_{2,t})$ with $R_{1,t} \leq R_{2,t} \leq 2R_{1,t}$. In this subsection, we argue that this method allows us to view $(X, Y)$ as hidden product characteristics. Additionally, factor loading parameters $\alpha$ and $\beta$ control the level of effects of $X$ and $Y$ on the product. This bridges the gap between quantitative modeling and experimental observations of purchasing behaviors of customers regarding price promotions. In fact, there are more than a million hits on Google with “BOGOF” as of Feb. 28, 2014; nevertheless, there has not been much attempts to explain it via quantitative modeling. However, one rare example is Thomas and Chrystal (2013) where the authors explain why “BOGOF” are so widespread using their relative utility pricing model.

According to Miracle (1965), product characteristics can be divided into as many as 9 categories according to which one can analyze the effects of different promotional strategies. More relevant to our context, Sinha and Smith (2000) focus on two features, namely consumption level and durability of a product. The level of consumption can be understood as a measure of the amount or the frequency of purchases. The higher the consumption level is, the more a customer is willing to pay for an additional item. This can be understood in our model as a high ratio of $R_{2,t}/R_{1,t}$. And we notice that it is effectively controlled by the ratio $c := \beta/\alpha$. On the other hand, the durability of a product is explicitly modeled by the parameter $\gamma$. One can easily imagine that the reservation prices would deteriorate quickly as the end of the sales horizon approaches for highly perishable products.

The interesting experiments conducted in Li et al. (2007) and Sinha and Smith (2000) test how customers respond to the promotional strategies “BOGOF” and “50% discount.” In principle, customers are better off by being offered 50% discounts for all purchased items as that provides more flexibility than the other promotion. However, it is not always so to the retailers. In the experiments by Li et al. (2007), customer preferences are compared across four product categories based on consumption level and durability. For example, one can consider powdered milk, fresh milk, powdered cheese, and yogurt. The main messages from the experiments are, first, preference differences decrease between two promotions as the consumption level increases, second, there is no statistically significant indication that the durability factor affects the difference of promotional preferences of customers. However, the preference difference is slightly more sensitive with respect to the durability factor (or stock-up level according to Li et al. (2007)) at a high consumption level than at a low consumption level.

Figure 1 illustrates the preference differences of customers with varying parameters. Here, preference difference is defined as the difference between purchase probabilities $\pi_{1,t} + \pi_{2,t}$ of “BOGOF” and “50% discount.” In the left panel, we observe that this difference decreases as $c$ increases. This is consistent with the first experimental observation described in the previous paragraph. In the right panel, $\gamma$ refers to the
durability factor so that it indicates that the products are more durable as \( \gamma \) increases. Then, for each fixed consumption level, we observe that the preference difference does not vary much compared to the difference induced by \( c \) at least for the parameter values considered. This is consistent with the second experimental observation. In the figure, \( X \) and \( Y \) are assumed to be bivariate normal.

4 Analysis of Promotional Strategies

Recall that we consider three policies \( \mathcal{A} = \{b, f, n\} \). We aim to understand the seller’s optimal choices at different combinations of the inventory level \( s \) and the remaining time \( t \). It is assumed that \( \gamma = 1 \) in this section. We start by analyzing the corresponding purchase probabilities and marginal revenues.

4.1 Purchase probabilities and marginal revenues

Since \( \gamma = 1 \), the purchase probabilities are independent of \( t \). It is then readily checkable that \( \pi_1^b = 0 \) and

\[
\begin{align*}
\pi_2^b &= P(\alpha X + 2\beta Y > p), \\
\pi_1^f &= P(\alpha X + \beta Y > p) > \beta Y), \\
\pi_2^f &= P(2\beta Y > p), \\
\pi_1^n &= P(\alpha X + \beta Y > p) > \beta Y), \\
\pi_2^n &= P(\beta Y > p).
\end{align*}
\]

For each \( a = (p, q) \in \mathcal{A} \), let us denote the associated total purchase probability and the marginal expected revenue by \( \Pi^a = \pi_1^a + \pi_2^a \) and \( \Lambda^a = p\pi_1^a + pq\pi_2^a \). These are important quantities that are directly related to customers’ purchasing behavior. As an illustration, Figure 2 shows how \( \Pi^b, \Pi^f, \Pi^n \) differ from each other. Obviously, \( \Pi^f \geq \Pi^b \geq \Pi^n \). The magnitude of such a difference depends on the probability distribution of \((\alpha X, \beta Y)\) over the shaded regions.

Figure 1: Preference differences between “BOGOF” versus “50% off”: \( X, Y \) are bivariate normal with mean 5, variance 1, correlation –0.7, and \( \alpha + \beta = 4 \).
In addition, simple but interesting behaviors can be shown by varying the model inputs such as $\alpha$, $\beta$ and $p$. For instance, we note that $R_2/R_1 = 1 + cY/(X + cY)$ with $c = \beta/\alpha$. It is clear that $\lim_{c \to 0} R_2/R_1 = 1$ and $\lim_{c \to \infty} R_2/R_1 = 2$. In other words, the higher (the smaller) $c$ is, the more (the less) customers value the second item on top of the first one. Such behaviors are found to depend on stock-up possibilities or consumption levels in the literature. In our model, we obtain the following asymptotic results. The expectations of $X, Y$ are denoted by $\mu_X$ and $\mu_Y$.

**Lemma 1** Suppose that $\gamma = 1$ and the expected value of $R_1$ is fixed at $\theta$. For $c = \beta/\alpha$, we have

1. as $c$ increases,
   \[
   \lim_{c \to \infty} \frac{\Pi^f}{\Pi^b} = 1, \quad \lim_{c \to \infty} \frac{\Pi^b}{\Pi^n} = \lim_{c \to \infty} \frac{\Pi^f}{\Pi^n} = \frac{P(Y > 0.5p \mu_Y/\theta)}{P(Y > p \mu_Y/\theta)};
   \]
   \[
   \lim_{c \to \infty} \frac{\Lambda^f}{\Lambda^b} = 1, \quad \lim_{c \to \infty} \frac{\Lambda^b}{\Lambda^n} = \lim_{c \to \infty} \frac{\Lambda^f}{\Lambda^n} = \frac{P(Y > 0.5p \mu_Y/\theta)}{2P(Y > p \mu_Y/\theta)};
   \]

2. as $c$ decreases,
   \[
   \lim_{c \to 0} \frac{\Pi^b}{\Pi^n} = 1, \quad \lim_{c \to 0} \frac{\Pi^f}{\Pi^b} = \lim_{c \to 0} \frac{\Pi^f}{\Pi^n} = \frac{P(X > 0.5p \mu_X/\theta)}{P(X > p \mu_X/\theta)};
   \]
   \[
   \lim_{c \to 0} \frac{\Lambda^b}{\Lambda^n} = 1, \quad \lim_{c \to 0} \frac{\Lambda^f}{\Lambda^b} = \lim_{c \to 0} \frac{\Lambda^f}{\Lambda^n} = \frac{P(X > 0.5p \mu_X/\theta)}{2P(X > p \mu_X/\theta)}.
   \]

The proof is omitted because it is straightforward from the observation that $\alpha \to 0$ and $\beta \to \theta/\mu_Y$ when $c$ increases to infinity and $\theta$ is fixed. The case of decreasing $c$ can be handled in a similar fashion.

Albeit simple, the above observations shed some light on relative characteristics of the strategies. If we regard purchase probabilities as one measure of product attractiveness, then “50% off” and “BOGOF” are close in that measure and dominant over “no promotion” at high level of $c$ whereas “50% off” is better than the others at low level of $c$. Hence, the parameter $c$ should not be too small for “BOGOF” to be attractive to customers compared to price discounts.

As shown in the next subsection, it is marginal revenue $\Lambda$ rather than $\Pi$ that affects optimal strategic choices for the seller. Therefore, it is desirable to have a good understanding of the properties of marginal revenues. From Lemma 1, it still holds that “50% off” and “BOGOF” are close in their marginal revenues at
high $c$, but they are not necessarily better than “no promotion.” The critical quantity in the first statement of the lemma is of form $0.5 F_Y(t) / F_Y(2t)$ where $F_Y$ is the complementary cumulative distribution function of $Y$. Thus, the role of the probability distribution of $Y$ becomes notable as illustrated in the examples below.

**Example 1** Suppose that $Y$ is a normal random variable truncated at 0 to ensure its nonnegativity. If the unit price $p$ is sufficiently high, then the marginal revenue of “50% off” or “BOGOF” is greater than that of “no promotion” at high $c$. This can be seen in the limit

$$\lim_{t \to \infty} \frac{0.5 \bar{F}_Y(t)}{\bar{F}_Y(2t)} = \lim_{t \to \infty} \frac{\exp \left( -\frac{(t - \mu_Y)}{2\sigma_Y^2} \right)}{4 \exp \left( -\frac{(2t - \mu_Y)^2}{2\sigma_Y^2} \right)} = \infty$$

where $\sigma_Y^2$ is the variance of $Y$.

**Example 2** An intermediate behavior can be obtained by assuming that $Y$ is regularly varying. If $Y$ has index $\rho \geq 0$, then by definition of regularly varying functions

$$\lim_{t \to \infty} \frac{\bar{F}_Y(at)}{\bar{F}_Y(t)} = a^{-\rho} \Rightarrow \lim_{t \to \infty} \frac{0.5 \bar{F}_Y(t)}{\bar{F}_Y(2t)} = 2^{\rho-1}.$$ Hence, the relative profitability of “50% off” or “BOGOF” depends on $\rho$ in this case.

The first example shows that $\Lambda^b$ and $\Lambda^f$ exceed $\Lambda^n$ if $c, p$ are sufficiently large and $Y$ is normally distributed. This is relevant to the case where we have $\min \{\Lambda^b, \Lambda^f\} > \Lambda^n$ in the next subsection. When $Y$ is regularly varying, we have a similar behavior if $\rho > 1$; however, its magnitude is smaller. We note that a similar argument applies to the low $c$ case as well by considering the variable $X$.

### 4.2 Properties of optimal strategies

Let us now consider the seller’s problem of choosing an optimal strategy in $\mathcal{A} = \{b, f, n\}$ to maximize the sales revenue. The optimality equation is given in Section 3. When the inventory $s = 1$, the seller cannot implement strategy $b$. Thus, we assume that the posted price is simply $p/2$. This leads to the optimality equation

$$V_t(1) = \lambda \pi' \frac{p}{2} + (1 - \lambda \pi') V_{t-1}(1) \Rightarrow V_t(1) = \frac{p}{2} \{1 - (1 - \lambda \pi')^{t-1}\}$$

where the purchase probability is $\pi'_1 = \mathbb{P}(R_1 > p/2)$, which is easily seen to be $\Pi'$. The next proposition partially characterizes optimal promotional strategies that the seller takes over the selling horizon.

**Proposition 1** With $\mathcal{A} = \{b, f, n\}$ and $\gamma = 1$, the following statements hold:

1. if $\Lambda^n > \Lambda^f$, then it is never optimal to choose strategy $f$;
2. if $\Lambda^n > \Lambda^b$, then it is never optimal to choose strategy $b$.

Consequently, if $\Lambda^n > \max \{\Lambda^f, \Lambda^b\}$, then it is always optimal to choose strategy $n$. 


It is helpful to explicitly write maximands in the optimality equations, say $V_t^a(s)$ for $a \in \mathcal{A}$:

$$
V_t^f(s) = \lambda \pi_1^f (p/2 + V_{t-1}(s-1)) + \lambda \pi_{12}^f (p + V_{t-1}(s-2)) + (1 - \lambda \pi_1^f - \lambda \pi_{12}^f)V_{t-1}(s),
$$

$$
V_t^b(s) = \lambda \pi_2^b (p + V_{t-1}(s-2)) + (1 - \lambda \pi_2^b)V_{t-1}(s),
$$

$$
V_t^n(s) = \lambda \pi_1^n (p + V_{t-1}(s-1)) + \lambda \pi_{12}^n (2p + V_{t-1}(s-2)) + (1 - \lambda \pi_1^n - \lambda \pi_{12}^n)V_{t-1}(s).
$$

For the first case, we note that the condition implies that $n$ or $b$ is optimal at $t = 1$ for any inventory level. For induction, we also note that

$$
V_t^n(s) - V_t^f(s) = \lambda p \left( \pi_1^n + 2\pi_{12}^n - 0.5\pi_1^f - \pi_{12}^f \right) + \lambda \left( \pi_1^n + \pi_{12}^n - \pi_1^f - \pi_{12}^f \right) \left\{ V_{t-1}(s) - V_{t-1}(s-1) \right\}
$$

$$
+ \lambda \left( \pi_2^n - \pi_{12}^n \right) \left\{ V_{t-1}(s-1) - V_{t-1}(s-2) \right\}.
$$

It is obvious that the value function $V_t(s)$ is increasing in $s$ at any time $t$. Collecting the conditions in $\Lambda^n > \Lambda^f$, $\Pi^f \geq \Pi^n$, and $\pi_2^n \geq \pi_{12}^n$, we can conclude that strategy $f$ is not optimal at time $t$. This completes the induction step.

For the second case, we have similarly as above

$$
V_t^n(s) - V_t^b(s) = \lambda p \left( \pi_1^n + 2\pi_{12}^n - \pi_{12}^b \right) + \lambda \left( \pi_2^n - \pi_1^n - \pi_{12}^n \right) \left\{ V_{t-1}(s) - V_{t-1}(s-1) \right\}
$$

$$
+ \lambda \left( \pi_2^n - \pi_{12}^n \right) \left\{ V_{t-1}(s-1) - V_{t-1}(s-2) \right\}.
$$

Then, the same argument with $\Lambda^n > \Lambda^b$, $\Pi^b \geq \Pi^n$, and $\pi_{12}^b \geq \pi_{12}^n$ yields the result.

The last statement is a trivial consequence of these two observations.

The above result permits us to consider the following remaining alternatives:

- $\mathcal{A} = \{a, n\}$ and $\Lambda^a > \Lambda^n$ where $a$ is either $b$ or $f$;
- $\mathcal{A} = \{b, f, n\}$ and $\min\{\Lambda^b, \Lambda^f\} > \Lambda^n$. 

Figure 3: Optimal strategies over the sales horizon: (a) $\mathcal{A} = \{f, n\}$, (b) $\mathcal{A} = \{b, n\}$.
The first case with \( a = f \) resembles the classical dynamic pricing problem as the seller sets the price at either \( p \) or \( p/2 \), but the important difference is that each customer is allowed to buy up to two items and thus any typical analytical approaches in the literature do not apply. However, the intuition behind this one and the classical model is the same. When \( t \) is close to the end of the horizon, it is optimal to offer a discount (or “BOGOF”); otherwise, “no promotion” is optimal if no pressure is on the seller to enhance the sales.

Figure 3 illustrates typical outcomes in the first case. Part (a) shows the optimal strategies among \( \{f, n\} \). Since \( \Lambda^f > \Lambda^n \), it is always optimal to take “50% off” at \( t = 1 \). As the time increases, there are more opportunities to raise revenues by setting the price high at \( p \). Therefore, “no promotion” becomes optimal if the remaining time is sufficiently large. The inventory level plays a role as well because higher revenues can be achieved from selling more items to customers by taking the price promotion \( f \). Quite similar behaviors are observed for the case \( A = \{b, n\} \) in part (b).

Unlike the first case, the second one has a new type of trade-off between “50% off” versus “BOGOF” that is different from the conventional dynamic pricing. We see that the insights in the previous paragraph are still true in this situation because it is best for the seller to set the price high as long as a full sale is highly probable. However, it is interesting how the strategies \( f \) and \( b \) play as the time and the inventory level vary. To focus on this interplay and trade-off and to avoid unnecessary complications that do not affect managerial insights, we analyze the case of \( A = \{f, b\} \) in the rest of this section.

When there are only two choices “50% off” or “BOGOF,” a few words can be said from our intuition. On the one hand, it is the former that is at least as good as the latter from the customer’s viewpoint (buying a second item is at the customer’s discretion). On the other hand, although the seller makes \( p \) from either strategy by selling two items, strategy \( b \) offers the seller another option to raise revenue especially when it is desirable to reduce the inventory at a faster rate because \( \pi_b^2 \geq \pi_f^2 \).

The proposition below characterizes the optimal strategic choices of the seller in the \((s, t)\) space. The first statement confirms the previous reasoning by showing that the region in which strategy \( b \) is optimal becomes larger and larger as \( t \) approaches the sales horizon and \( s \) is greater. However, such an opportunity disappears if “50% off” has a higher expected revenue than “BOGOF.”

**Proposition 2** With \( A = \{b, f\} \) and \( \gamma = 1 \), the following statements hold:

1. if \( \Lambda^b > \Lambda^f \) and if strategy \( b \) is optimal for \( t = \tau, \tau - 1, \ldots, 1 \) and at \( s = \varsigma - 1, \varsigma - 2 \) with \( \varsigma \geq 4 \), then it is optimal to choose strategy \( b \) for \( t = \tau + 1, \ldots, 1 \) and at \( s = \varsigma \);

2. if \( \Lambda^f > \Lambda^b \), then it is always optimal to choose strategy \( f \).

Its proof is based on induction and the initiation step is proved in the next lemma. Proofs of both claims are deferred to the appendix.

**Lemma 2** Suppose that \( A = \{b, f\}, \gamma = 1, \) and \( s = 2 \). If \( \Lambda^b > \Lambda^f \), then there exists a finite \( \tau \geq 1 \) such that it is optimal to take strategy \( f \) for \( t = T, T - 1, \ldots, \tau + 1 \) and to switch to strategy \( b \) from \( t = \tau \) to \( t = 1 \). If \( \Lambda^f > \Lambda^b \), then it is always optimal to take strategy \( f \).

An interesting implication of two propositions is that it is possible to deduce the retailer’s optimal actions as \( t \) gets close to the end of sales horizon.
Corollary 1 Suppose that the same policy is optimal at the inventory level 2 and 3 for \( t = 1, 2, \ldots, \tau \). Then, the policy is still optimal in the region
\[
\left\{ (s, t) | s = 2k, t \leq \tau + k - 1 \text{ or } s = 2k + 1, t \leq \tau + k - 1 \text{ where } k = 1, 2, \ldots \right\}.
\]

Proof: We first note that if \( \Lambda^b = \Lambda^f \), then all the arguments above are still valid. Next, consider the case \( \Lambda^b > \Lambda^f \). When \( k = 1 \), the statement is true by assumption. Suppose it is true up to \( k \leq k_0 \). Then, at the inventory levels \( 2k_0 \) and \( 2k_0 + 1 \) the policy is optimal up to \( t = \tau + k_0 - 1 \). This implies that at the inventory level \( 2(k_0 + 1) \), it is optimal up to \( t = \tau + k_0 \) by Proposition 2. We can apply the same reasoning for the inventory levels \( 2k_0 + 1 \) and \( 2(k_0 + 1) \). Then the same is true at level \( 2(k_0 + 1) + 1 \) up to \( t = \tau + k_0 \). The induction step is done. The other case \( \Lambda^b < \Lambda^f \) is similar hence the proof is omitted.

We note that regardless of the condition on the purchase probabilities the same policy is applied for all \( s \) at \( t = 1 \). Hence, \( \tau \geq 1 \). An immediate consequence of this fact is that a single policy dominates in the region
\[
\bigcup_{k=1}^{\infty} \left\{ (s, t) | s = 2k \text{ or } s = 2k + 1, t = 1, 2, \ldots, k \right\}.
\]

In order to illustrate our findings, we present numerically found optimal strategies in Figure 4. Lightly shaded regions in the first two panels show the regions in which “BOGOF” is more profitable than “50% off.” Here, \( \delta = \frac{\pi^f_1}{(\pi^b_2 - \pi^f_2)} \) and this turns out to be a convenient measure to compare two strategies. Based on extensive numerical experiments, it is observed that optimal strategies with the same \( \delta \) value exhibit qualitatively similar behaviors. For instance, \( 2\pi^b_2 \) is greater than \( \pi^f_1 + 2\pi^f_2 \) if and only if \( \delta \in (1, 2) \); if \( \delta > 2 \), then “50% off” is always optimal. And when \( \delta \in (1, 2) \) so that \( \Lambda^b > \Lambda^f \), the shapes of optimal strategies look similar to those in Figure 4. On the other hand, two strategies offer similar marginal revenues as \( \delta \) converges to 2, and the region for “50% off” is expanded as \( \delta \) increases to 2. This behavior is illustrated in part (c) which shows the values of \( \delta \) at which the optimal strategy at a given \((s, t)\) changes from “BOGOF” to “50% off.” As expected, the seller is quick in changing optimal strategies in the lower left corner of the region as the relative benefit of \( b \) decreases.

Remark 2 One last comment is that the region found above can be enlarged by imposing a little more stringent condition on the purchase probabilities. For instance, suppose \( 2\pi^b_2 > \pi^f_1 + 3\pi^f_2 \). Then, it can be
shown that strategy $b$ is optimal for $t = 1, \ldots, \tau + 1$ if it is so for $t = 1, \ldots, \tau$ at $s = 2$. Then, a similar argument as Corollary 1 yields a larger set.

5 Model Implementation and Computational Results

5.1 Modeling dependencies

The dynamic programming (1) in the generalized model is solved by backward recursion with boundary conditions $V_0(s) = V_t(0) = 0$ for all $s$ and $t$. The most important factor that affects optimal strategic choices is the purchase probabilities $\pi_{1,t}^a$ and $\pi_{2,t}^a$ for $a = (p, q) \in A$. A widely used and flexible approach to modeling correlated random variables is the method of copulas.

A copula is a multivariate distribution function of random variables whose supports are the unit interval $[0, 1]$. It has been a very useful tool as it provides a way of handling the dependence structure between random variables via Sklar’s Theorem. It states that for a given joint multivariate distribution and corresponding marginal distributions, there is a copula function, say $C$, that relates the joint and marginal distributions. In 2-dimensional case, for given random variables $X, Y$, their joint distribution can be written as

$$P(X \leq x, Y \leq y) = C(F_X(x), F_Y(y)),$$

where $F_X, F_Y$ are the marginal distribution functions of $X$ and $Y$, respectively, and $C$ is the joint distribution function of two correlated uniform random variables $U = F_X(X)$ and $V = F_Y(Y)$.

In our experiments, we assume that the marginal distributions of $X, Y$ are normal distributions with suitably chosen means and variances so that $P(X < 0$ or $Y < 0)$ is close to zero. Since this makes each $R_i$ normally distributed and a normal distribution is light-tailed, this models the situation in which the reservation prices quickly diminish as the price hikes. We however note that copulas are flexible enough to handle much more general cases. The adoption of copulas enables us to change the way $R_i$’s depend on each other and to test the effects of such dependence structure on the top of the usual Pearson correlation coefficient.

It is known that any copula function $C$ has upper and lower bounds which are called Fréchet-Hoeffding bounds. The following examples illustrate how one can compute $\pi_{1,t}$’s using $C$ and examine the extreme cases of those bounds.

**Remark 3** Suppose that $X, Y$ have probability densities $f_X, f_Y$, respectively. We further assume that $C$ is differentiable almost everywhere and $\alpha = \beta = \gamma = 1$ for simplicity. Then, for $a = (p, q) \in A$, it can be shown that

$$\pi_{1} = P(X + Y > p, X + Y - p > X + 2Y - p(1 + q)) = \int_0^{pq} P(X > p - y | Y = y) f_X(y) dy$$

$$= F_Y(pq) - \int_0^{pq} C_2(F_X(p - y), F_Y(y)) f_Y(y) dy,$$
\[ \pi_2^a = F_Y(pq) - \int_{pq}^{p(1+q)/2} C_2(F_X(p(1+q) - 2y), F_Y(y)) f_Y(y) dy \]

where \( C_2 \) is the derivative of \( C \) with respect to the second argument. Here we used the fact that \( P(U \leq u | V = v) = C_2(u, v) \).

**Example 3** In the remark above, we additionally impose \( C(u, v) = \min\{u, v\} \) which is the Fréchet-Hoeffding upper bound. In this case, \( X, Y \) become comonotonic random variables. Then, \( C_2(u, v) = \mathbb{1}_{\{u > v\}} \).

For strategy \( b = (p, 0) \) and \( f = (p/2, 1) \), it is readily found that

\[ \Pi^b = F_Y(y^*), \quad \Pi^f = F_Y(y^{**}) \]

where \( y^* > y^{**} \) are points in \( (0, p/2) \) such that \( F_X(p - 2y) = F_Y(y) \) and \( F_X(p/2 - y) = F_Y(y) \), respectively. One can compute \( \pi_1^a \)’s in a similar way. This observation leads us to

\[ \Lambda^b > \Lambda^f \Leftrightarrow P(y^{**} < Y \leq y^*) < P(y^* < Y \leq p/2). \]

**Example 4** In the other extreme case, we consider the Fréchet-Hoeffding lower bound \( C(u, v) = (u + v - 1)^+ \). This choice makes \( X, Y \) counter-monotonic. Since \( C_2(u, v) = \mathbb{1}_{\{u + v > 1\}} \), straightforward computations yield

\[ \Pi^b = P(Y \in D^*), \quad \Pi^f = P(Y \in D^{**}) \]

where \( D^* \subset D^{**} \) are the complements of the sets \( \{y \in [0, p/2] | F_X(p - 2y) + F_Y(y) > 1\} \) and \( \{y \in [0, p/2] | F_X(p/2 - y) + F_Y(y) > 1\} \), respectively. Then, it is not difficult to see that

\[ \Lambda^b > \Lambda^f \Leftrightarrow P(Y \in D^{**} \setminus D^*) < P(Y \in D^* \cap [0, p/2]). \]

### 5.2 Numerical results for optimal strategies

*Purchase probabilities and optimal strategies.* Previously, we analyzed the optimal strategic choices of the seller by focusing on the trade-off between “BOGOF” and “50% off.” In this subsection we shall strengthen our understanding of optimal strategies through several numerical experiments. Let us continue to assume \( \gamma = 1 \). The remaining cases that need further investigation are \( \min\{\Lambda^b, \Lambda^f\} > \Lambda^n \). See Section 4.2 for a detailed description of optimal strategies in other cases.

Figure 5 shows the optimal promotional strategies over the sales horizon with \( T = 100 \) under different purchase probability settings but with the constraint \( \Lambda^b > \Lambda^f > \Lambda^n \). Part (a) of Figure 5 is similar to the
Lastly, we also see that the region for strategy $\Lambda^b$ diminishes as $\Lambda^b$ increases. This coincides with our intuition that the seller chooses a high price if the sales horizon is sufficiently long.

A more interesting behavior is shown in the lower row of Figure 5. In this case, the probability $\Pi^f$ is fixed at 0.4 but we vary the composition of $\pi^f_i$’s and the parameter $\delta$. Panels (d) and (f) are similar to Figure 4, except that some regions are taken by strategy $n$, where the region of strategy $f$ is larger when $\delta$ is close to 2. In part (e), we observe that “50% off” appears in some small areas when the inventory level $s$ is an odd integer. This can be understood as the effect of the subtle trade-off between “BOGOF” and “50% off”; for instance, even if strategy $b$ is optimal at $(s, t) = (2, t_0)$, strategy $f$ could be beneficial at $(s, t) = (3, t_0)$ due to its flexibility. However, such an advantage quickly disappears as $t$ decreases or $s$ increases.

Regarding the case of $\Lambda^f > \Lambda^b > \Lambda^n$, see Figure 6. In those three panels, “BOGOF” is not present. This is not at all surprising because, according to Proposition 2, strategy $f$ dominates strategy $b$ if $\Lambda^f > \Lambda^b$. This implies that the region for strategy $b$ would be taken over by strategy $f$ if we add $f$ to $\mathcal{A}$ in Figure 3(b). Lastly, we also see that the region for strategy $f$ diminishes as $\pi^f_i$’s increase. This is because the seller can postpone price promotion because larger $\pi^f_i$’s mean better sales opportunities once the promotion starts.

Effects of model parameters. We next study how the optimal strategies depend on some important model parameters: price $p$, “consumption level” $c$, and time depreciation factor $\gamma$. First, as shown in Figure 7, strategy $b$ becomes more attractive as $c$ increases at a high price. This is because the reservation price $R_2$ is likely to have larger values whereas $E[R_1] = \alpha \mu_X + \beta \mu_Y$ is fixed at 20. Here $X, Y$ are normally distributed
Figure 6: Optimal strategies over the sales horizon where $\Lambda^f > \Lambda^b > \Lambda^n$, $(\pi^f_1, \pi^f_2) = (0.2, 0.2)$ and $(\pi^n_1, \pi^n_2) = (0.1, 0.05)$ and $\delta = 5$.

Figure 7: Optimal strategies with respect to model parameter $c$: $X, Y \sim N(5, 1)$ and Gaussian copula with $\rho = 0.5$, $\alpha + \beta = 4$, $\gamma = 1$ and $c = \beta/\alpha$. 
with mean 5 and variance 1. A Gaussian copula $C$ is given by

$$C(u, v) = \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$$

where $\Phi(\cdot)$, $\Phi_2(\cdot, \cdot)$ are the cumulative distribution functions of univariate and bivariate normal random variables (with correlation $\rho$), respectively. In contrast, strategy $f$ dominates for small $c$ values at a high price, which is implied by Lemma 1. We note, however, that these behaviors are not seen at a small price $p$ where purchase probabilities without any promotion are already large enough.

Second, Figure 8 compares optimal strategies when different values for the depreciation factor $\gamma$ are used. Notice that the effect of a smaller $\gamma$ is somewhat different from the effect of a larger $\rho$ because the effect of the former on the purchase probabilities escalates over time. As shown in the figure, “50% off” could enter the picture at $\gamma = 0.99$ although there is no $f$ when $\gamma = 1$. This is due to the quickly decreasing purchase probabilities as $t$ gets closer to zero. Most notably, the boundary between strategies $b$ and $f$ is now nonlinear unlike previous figures.

**Effects of dependence structure.** The bivariate random vector $(X, Y)$ is the key ingredient of our modeling approach. And its distributional properties determine the reservation prices $R_i$’s and eventually the purchase probabilities and marginal expected revenues. To better understand the effect of the dependence structure of $(X, Y)$, we first assume that $(X, Y)$ is bivariate normal and see the optimal strategies as the correlation coefficient $\rho$ varies in Figure 9.

In the upper row of the figure, scatter plots of sampled $(R_1, R_2)$ values are given for $\rho = -0.99, -0.5$ and 0. The two dotted lines represent the extreme cases of $R_2 = 2R_1$ and $R_2 = R_1$, respectively. The panels in the lower row show that the optimal strategies can change drastically as $\rho$ changes (all other parameters fixed) and this eventually leads to quite different optimal expected revenues as well. In this particular example, $\pi_i^b$’s are large enough and thus strategy $f$ is nowhere optimal for any $\rho \geq 0$. However, as $\rho$ becomes more negative, one can easily check that the variances of $R_i$’s become smaller and this yields smaller purchase probabilities because $p = 25 > 20 = \mathbb{E}[R_1]$. This decreasing $\rho$ eventually makes strategy $f$ optimal in some regions as shown in parts (a) and (b) of Figure 9.
While the Pearson correlation coefficient $\rho$ tells us a lot about the behaviors of the reservation prices and optimal choices, a more subtle dependence structure can be considered. As a final remark in this subsection, let us compare the optimal expected revenues if two other copula functions are incorporated; namely, Clayton and Gumbel:

\[
C(u, v) = \begin{cases} 
\max \left( u^{-\theta_{\text{clayton}}} + v^{-\theta_{\text{clayton}}} - 1, 0 \right)^{-1/\theta_{\text{clayton}}}, & \theta_{\text{clayton}} \in (-1, \infty) \setminus \{0\}; \\
\exp \left\{ - \left[ (- \ln u)^{\theta_{\text{gumbel}}} + (- \ln v)^{\theta_{\text{gumbel}}} \right]^{1/\theta_{\text{gumbel}}} \right\}, & \theta_{\text{gumbel}} \in [1, \infty). 
\end{cases}
\]

They are known to exhibit different tail behaviors. We first match their correlation coefficient via the relationships

\[
\rho = \sin \left( \frac{\pi \tau}{2} \right), \quad \tau = \frac{2\theta_{\text{clayton}}}{1 - \theta_{\text{clayton}}}, \quad \tau = \frac{1}{1 - \theta_{\text{gumbel}}}
\]

where $\tau$ is Kendall’s tau. We refer the reader to Nelson (2006) for more information about copulas.

Then, optimal expected revenues are compared under three different parameter settings in Figure 10. More precisely, the $y$-axis represents the ratio of the optimal expected revenues from Clayton copula and Gumbel copula and we note that a different dependence structure can yield up to 3% difference in addition to the effect of linear correlation.
Figure 10: Ratio of optimal expected revenues with $\text{RD} = \text{revenue(Clayton)}/\text{revenue(Gumbel)}$: $X, Y \sim \text{N}(5, 1)$ and $\gamma = 1, c = \beta/\alpha$

6 Extended Model

So far we have restricted our attention to the case $\mathcal{A} = \{b, f, n\}$. In this section, we consider a generalized version of the proposed model by enlarging the set of admissible strategies $\mathcal{A} = \mathcal{P} \times \mathcal{Q}$ for some sets of prices and discounts $\mathcal{P}$ and $\mathcal{Q}$. Actually, the benefits of dynamic pricing or discounting are widely known. However, the interplay between pricing and discounting makes the model not amenable to mathematical analysis. Several experiments are conducted in order to quantify the benefits of dynamic nonlinear pricing, compared to other strategies such as static pricing, dynamic pricing, etc.

Remark 4 As mentioned in the introduction, Chen and Riordan (2013) derive conditions on the dependence structure of reservation prices under which a small discounting is beneficial for the seller of product bundles. In the same spirit, we can also derive a condition on $(X, Y)$ such that the marginal expected revenue increases with a small fraction of discounting.

Suppose that $X, Y$ have continuous distributions $F_X, F_Y$ with densities $f_X, f_Y$. We further assume that $C$ has second order derivatives and that $\alpha = \beta = \gamma = 1$. Then, there exists $q < 1$ such that $\Lambda^{(p, q)} > \Lambda^{(p, 1)}$ if and only if

$$ph(p) > 1$$

where $h(x) = f_Y(x)/F_Y(x)$ is the hazard rate function of $Y$. This implies that the seller can improve the sales via discounting (in addition to optimal pricing) at least near the end of the sales horizon if the above condition is satisfied.

Model setting and preliminary analysis. For our numerical experiments, we consider a bivariate normal for $(X, Y)$ with $\rho = 0.9$ and set the inventory level at 20. The set of admissible strategies is given by

$$\mathcal{A} = \mathcal{P} \times \mathcal{Q}, \quad \mathcal{P} = \{8, 9, \ldots, 40\}, \quad \mathcal{Q} = \{1\%, 2\%, \ldots, 100\%\}. \quad (2)$$

Figure 11 shows optimal prices and discounts as functions of time-to-maturity. As one can expect, optimal prices and discount rates tend to decrease as the time-to-maturity decreases, but with some fluctuating behaviors near the end of sales horizon. This finding can be explained by the relationship between price and discount rate; often price reduction and small $q$ induce similar effects on customers’ purchase behaviors.
and that results in subtle differences for optimal decisions. Such dynamics are more visible for small time-to-maturities. Consequently, the monotonicity of price or discount rate breaks down. We further compare optimal values according to the “consumption level” \( c \). Optimal prices in both cases are formed at similar levels while there is an evident difference in the movements of discount rates. Notably, the discount rate at a low \( c \) is shaped lower overall. This coincides with our intuition that small \( q \) should be offered to induce purchases if \( R_2 \) is relatively low.

**Effects of adaptive thinning.** It is typical to assume different customer pools for different purchase sizes. If we assume Poisson arrivals for instance, then this means that such different customer arrivals are formed by fixed thinning probabilities. In contrast, the proposed model assumes that thinning happens adaptively according to the given menu \((p, q)\). For a better understanding of this feature, we compare the optimal revenues from the proposed model and the model analyzed in Levin and Nediak (2014).

To be specific, let us denote the arrival probabilities of customers per period for one item and two items by \( \lambda_1 \) and \( \lambda_2 \), respectively. For simplicity, we set \( \gamma = 1 \). Then, the optimality equation becomes

\[
V_t(s) = \max_{(p, q) \in A} \left\{ \lambda_1 \tilde{\pi}_1(p + V_{t-1}(s-1)) + \lambda_2 \tilde{\pi}_2(p + pq + V_{t-1}(s-2)) + (1 - \lambda_1 \tilde{\pi}_1 - \lambda_2 \tilde{\pi}_2)V_{t-1}(s) \right\},
\]

where \( \tilde{\pi}_1 = P(\alpha X + \beta Y > p) \) and \( \tilde{\pi}_2 = P(\alpha X + 2\beta Y > p + pq) \). For comparison, we apply strategies from the above equation to our modeling framework and then compute the expected revenue at each decision epoch \((s, t)\). We repeat the same procedure for different combinations of \((\lambda_1, \lambda_2)\) such that \( \lambda_1 + \lambda_2 = \lambda \). The average difference of revenues is given in Figure 12. According to this experiment, we observe that adaptive thinning makes a larger difference if the inventory size or remaining time is large, under the assumption that customer’s purchase sizes are dependent on the posted menu.

**Comparison with other sales strategies.** We lastly examine how much improvements in profits can be brought by the joint consideration of dynamic pricing and discounting. For that matter, we generate total 108 different problem instances, and for each instance we identify optimal prices and discounts over a 100-period horizon under various pricing and discounting schemes. More specifically, the set \{1, 2, 3\} is considered for possible values of \( \alpha, \beta \), and the time depreciation factor \( \gamma \) is assumed to take a value in \{0.7, 0.8, 0.9, 1\}. Also, three dependence structures are incorporated, namely Clayton, Gumbel and Gaussian.

Figure 11: Optimal price and discount rate in remaining time with inventory level 20, \( X, Y \sim N(5, 1) \) and Gaussian copula with \( \rho = 0.9, \alpha + \beta = 4, \gamma = 1 \) and \( c = \beta / \alpha \).
Figure 12: Average differences of expected revenues resulting from adaptive thinning.

Table 1: Comparison of optimal revenues compared to the benchmark case of dynamic pricing with discount rates: $P = \{8, 9, \ldots, 40\}$ and $Q = \{0\%, 1\%, \ldots, 100\\%\}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Average OG</th>
<th>Max OG</th>
<th>Min OG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic programming + Discount rate{0,1}</td>
<td>2.11%</td>
<td>2.96%</td>
<td>0.58%</td>
</tr>
<tr>
<td>$P = {8, 9, \ldots, 40}$, $Q = {0, 1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dynamic programming + No discounting</td>
<td>8.72%</td>
<td>11.14%</td>
<td>1.94%</td>
</tr>
<tr>
<td>$P = {8, 9, \ldots, 40}$, $Q = {1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monotonic constraint + Discount rate{0,1}</td>
<td>0.04%</td>
<td>0.07%</td>
<td>0.02%</td>
</tr>
<tr>
<td>$P = {8, 9, \ldots, 40}$, $Q = {0%, 1%, \ldots, 100%}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monotonic constraint + Discount rate{0,1}</td>
<td>5.95%</td>
<td>10.56%</td>
<td>1.94%</td>
</tr>
<tr>
<td>$P = {8, 9, \ldots, 40}$, $Q = {0, 1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed Price + No discounting</td>
<td>36.09%</td>
<td>54%</td>
<td>27.39%</td>
</tr>
<tr>
<td>$P = {20}$, $Q = {1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

There are six pricing and discounting schemes that we consider. The benchmark case is the case where we apply fully dynamic pricing and discounting together. The action space is set as in (2). Then, we compare the resulting expected revenue with other cases where we apply fully dynamic pricing with “BOGOF” only or dynamic pricing without any discounting. See the first two rows in Table 1. Three columns in the table report the average optimality gap (OG), the maximum OG, and the minimum OG among 108 problem instances, respectively. We notice that “BOGOF” induces 6% increase on average compared to the no discounting case but has 2% difference compared to the fully dynamic case. To be more realistic about price changes, we experimented with the constraint that prices are not allowed to increase in time. As shown in the third and fourth rows of Table 1, such a constraint turns out not to affect expected revenue much, making it a reasonable pricing and discounting strategy. The last row in the table reports the static pricing case, and a large difference in revenues up to 54% is observed.
7 Concluding Remarks

We studied a dynamic pricing problem of a seller who utilizes price promotional schemes in a finite horizon. A specific focus was given to “buy one get one free” promotion and “50% off” discount promotion. Based on the reservation price approach, we transformed the two-dimensional vector of reservation prices into a vector \((X, Y)\) in the first quadrant for modeling convenience. This approach provides a straightforward yet flexible modeling framework for reservation prices with certain constraints. We then studied the optimal strategic choices of the seller mathematically and numerically.

Important managerial insights are, first, “BOGOF” is profitable near the end of the sales horizon as long as its marginal expected revenue exceeds those of other schemes; otherwise, “no promotion” or “50% off” tend to dominate. Second, such a tendency is more visible for larger inventories and larger differences in marginal expected revenues. Additional observations have been made through numerical experiments. Optimal strategic choices were found to depend on model parameters such as \(c\), which was interpreted as consumption level, the posted price \(p\), and the time depreciation factor \(\gamma\). Another interesting observation was made by studying the impact of the dependence structure of \(X\) and \(Y\). The first order effect was examined by varying their linear correlation. We additionally noticed that there is a more subtle, second order but non-negligible effect induced by different copula structures.

Extended numerical experiments provided a better understanding of the model. We first quantified the effect of adaptive thinning. It was shown that higher revenues can be gained by incorporating the fact that a customer’s purchase size is affected by a posted menu. Second, more general promotional schemes such as “buy one get one with discount” were analyzed as well. Total 108 test cases were generated and their corresponding dynamic programs were solved. Although dynamic pricing is well known to increase the revenue, the additional increase thanks to dynamic discounting was more than 8% on average. Actually dynamic implementation of simply “buy one get one free” induced more than 6% increase in revenue compared to dynamic pricing with no discounting. This certainly confirms the popularity of “BOGO” strategies, but its effectiveness depends on several factors such as the dependence structure of product features, the remaining sales horizon and the inventory level.

There still remains a large room for improvement. Despite the fact that the reservation price approach has been widely accepted in the literature, empirical studies are required to make the proposed model implementable in practice. This is one limitation of the present work. Understanding product characteristics and the dependence structure of reservation prices would be particularly intriguing and challenging. Another limitation is that we gave only partial analytical results. It is beyond the scope of the present work but it would be quite interesting to analyze complete structural properties of optimal pricing and discounting as well as the value function possibly in a more general setting. A different direction for further development can be mentioned by noting that the current practices of dynamic pricing and promotions are diverse and widespread. For example, it is quite common to observe discounted offers as well as “buy one get one free” or “buy one get two free” for many different (but possibly highly correlated) products in the same store. However, a systemic approach to such diversified products and promotional schemes has yet to come. Lastly, the current investigation can be extended to incorporate consumers’ strategic acts when the seller is known to implement certain promotional strategies.
Acknowledgement

The authors would like to thank Prof. Tim Huh and three anonymous reviewers for their helpful comments. Kim’s work was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education (NRF-2014R1A1A2054868).

References


Salvi, P. 2013. Effectiveness of sales promotional tools: A study on discount, price off and buy one get one free offers in branded apparel retail industry in Gujarat Working paper.


**Appendix**

**Proof of Lemma 2:** Suppose $\Lambda^b > \Lambda^f$. This assumption implies that strategy $b$ is optimal at $t = 1$ because $V_1(2) = \max\{\lambda \pi_2^b p, \lambda(\pi_1^f p/2 + \pi_2^f p)\}$. Assume that strategy $b$ is optimal for $t = 1, 2, \ldots, \tau$. Then, the optimality equation becomes

$$V_t(2) = \lambda \pi_2^b p + (1 - \lambda \pi_2^b) V_{t-1}(2) \Rightarrow V_t(2) - p = (1 - \lambda \pi_2^b)(V_{t-1}(2) - p).$$
This yields $V_\tau(2) = p - p(1 - \lambda \pi_2^b)^\tau$. The same equation also implies that strategy $f$ is optimal at $t = \tau + 1$ if and only if

$$
\pi_2^b(p - V_\tau(2)) < \pi_1 f \left( \frac{p}{2} - V_\tau(2) + V_\tau(1) \right) + \pi_2^f(p - V_\tau(2))
\Leftrightarrow \left( \Pi^f - \pi_2^b \right) \frac{V_\tau(2)}{p} < \frac{1}{2} \pi_1 f + \pi_2^f - \pi_2^b + \pi_1 f V_\tau(1) p
\Leftrightarrow \frac{1}{2} \pi_1 f \varphi^\tau < (\Pi^f - \pi_2^b) \left( 1 - \frac{1}{p} V_\tau(2) \right)
$$

(3)

where $\varphi = 1 - \lambda \Pi^f$. With $\eta = 1 - \lambda \pi_2^b$, (3) is simplified to

$$
\frac{\pi_1 f}{2(\Pi^f - \pi_2^b)} < \left( \frac{\eta}{\varphi} \right)^\tau.
$$

Since $\eta > \varphi$, this inequality holds for a sufficiently large $\tau$.

Suppose that it is optimal to choose strategy $f$ for $t = \tau + 1, \ldots, \tau'$. The same policy would be optimal if (3) still holds at $t = \tau' + 1$ with $\tau$ replaced by $\tau'$. Direct computations from the optimality equation give us the dynamics of $V_{\tau}(2)$ so that we obtain

$$
\frac{1}{p} V_{\tau'}(2) = \frac{\varphi}{p} V_{\tau'-1}(2) + \lambda \Pi^f - \frac{\lambda \pi_1 f}{2} \varphi^{\tau' - 1}.
$$

To check (3), we proceed as follows:

\[
\begin{align*}
(\Pi^f - \pi_2^b) \left( 1 - \frac{1}{p} V_{\tau'}(2) \right) &- \frac{1}{2} \pi_1 f \varphi^{\tau'} \\
&= (\Pi^f - \pi_2^b) \varphi \left( 1 - \frac{1}{p} V_{\tau'-1}(2) \right) + \frac{1}{2} \pi_1 f \varphi^{\tau'-1} \left( \lambda (\Pi^f - \pi_2^b) - \varphi \right) \\
&> \frac{1}{2} \pi_1 f \varphi^{\tau'-1} \lambda (\Pi^f - \pi_2^b) \\
&= \frac{1}{2} \pi_1 f \varphi^{\tau'-1} \lambda (\Pi^f - \pi_2^b)
\end{align*}
\]

which is clearly positive. Inductively, we conclude that, for all $t > \tau$, policy $f$ is optimal.

Now suppose $\Lambda^b < \Lambda^f$. It is then optimal to take strategy $f$ at $t = 1$. Assume that the same strategy is optimal for $t = 1, 2, \ldots, \tau$. Then the optimality equation reads

$$
V_t(2) = \lambda \pi_1^f \left( \frac{p}{2} + V_{t-1}(1) \right) + \lambda \pi_2^f p + \left( 1 - \lambda \pi_1^f - \lambda \pi_2^f \right) V_{t-1}(2), \quad t = 1, 2, \ldots, \tau
$$

from which it follows that

$$
V_t(2) - p = \left( 1 - \lambda \Pi^f \right) (V_{t-1}(2) - p) + \lambda \pi_1^f V_{t-1}(1) - \frac{p}{2}
\Leftrightarrow \left( 1 - \lambda \Pi^f \right) (V_{t-1}(2) - p) - \lambda \pi_1^f \frac{p}{2} (1 - \lambda \Pi^f)^{t-1}.
$$

This is again simplified to

$$
V_t(2) = p \left( 1 - \varphi^t \right) - \frac{p}{2} \lambda \pi_1^f \varphi^{t-1}.
$$

25
Then, strategy $b$ is better than the other if and only if

\[
\frac{1}{2} \pi_1^f \varphi^r < \left( \Pi^f - \pi_2^b \right) \left( \varphi^r + \frac{1}{2} \lambda \pi_1^f \varphi^{r-1} \right).
\]

Since $\Lambda^f - \Lambda^b > 0$, this inequality always holds. Hence, strategy $f$ is optimal in this case. \qed

**Proof of Proposition 2: Part I**

The assumption $\Lambda^b > \Lambda^f$ implies that strategy $b$ is optimal for any inventory level at $t = 1$. We first record the condition that one prefers strategy $b$ to strategy $f$ at time $t$ and at the inventory level $x \geq 2$. From the optimality equation, it is easy to see that

\[
V_t(x) = \max \left\{ \lambda \pi_2^b (p - \Delta_2 V_{t-1}(x)), \lambda \pi_1^f \left( \frac{p}{2} - \Delta_1 V_{t-1}(x) \right) + \lambda \pi_2^f (p - \Delta_2 V_{t-1}(x)) \right\} + V_{t-1}(x).
\]

Then, strategy $b$ is better than the other if and only if

\[
\pi_2^b (p - \Delta_2 V_{t-1}(x)) > \pi_1^f \left( \frac{p}{2} - \Delta_1 V_{t-1}(x) \right) + \pi_2^f (p - \Delta_2 V_{t-1}(x)) \]

\[
\Leftrightarrow (p - \Delta_2 V_{t-1}(x)) > \delta \left( \frac{p}{2} - \Delta_1 V_{t-1}(x) \right) \tag{4}
\]

where $\delta$ is defined as $\pi_1^f / (\pi_2^b - \pi_2^f)$. We note that $\delta \in (1, 2)$.

Now suppose that strategy $b$ is optimal for $t = 1, 2, \ldots, \tau$ when the inventory level is at $s = 2$ and that the same is true for inventory level $s - 1$ and $s$. Then, the optimality equation is

\[
V_t(s) = \lambda \pi_2^b (p + V_{t-1}(s - 2)) + (1 - \lambda \pi_2^b) V_{t-1}(s), \quad t = 1, 2, \ldots, \tau.
\]

Re-writing this as $V_t(s) - p = (1 - \lambda \pi_2^b)(V_{t-1}(s) - p) + \lambda \pi_2^b V_{t-1}(s - 2)$ and using $V_1(s) = \lambda \pi_2^b p$, we get

\[
\sum_{j=0}^{t-1} (1 - \lambda \pi_2^b)^j (V_{t-j}(s) - p) = \sum_{j=0}^{t-1} (1 - \lambda \pi_2^b)^j V_{t-j}(s) + \sum_{j=0}^{t-1} (1 - \lambda \pi_2^b)^j \lambda \pi_2^b V_{t-j}(s - 2)
\]

\[
\Leftrightarrow V_t(s) = p - p \left( 1 - \lambda \pi_2^b \right)^t + \sum_{j=0}^{t-2} (1 - \lambda \pi_2^b)^j \lambda \pi_2^b V_{t-j}(s - 2).
\]

The same formula holds for $s - 1$ and for $s - 2$. Then, for $t = \tau$ we take the difference between two resulting equations from $s$ and $s - 2$:

\[
p - \Delta_2 V_\tau(s) = p - \sum_{j=0}^{\tau-2} (1 - \lambda \pi_2^b)^j \lambda \pi_2^b \Delta_2 V_{\tau-j}(s - 2)
\]

\[
= p \left( 1 - \lambda \pi_2^b \right)^{\tau-1} + \sum_{j=0}^{\tau-2} (1 - \lambda \pi_2^b)^j \lambda \pi_2^b (p - \Delta_2 V_{\tau-j}(s - 2))
\]

\[
> p \left( 1 - \lambda \pi_2^b \right)^{\tau-1} + \sum_{j=0}^{\tau-2} (1 - \lambda \pi_2^b)^j \lambda \pi_2^b \delta \left( \frac{p}{2} - \Delta_1 V_{\tau-j}(s - 2) \right)
\]

for $t = 1, 2, \ldots, \tau$. Next, we check the optimality of strategy $f$, (3), at $t = \tau + 1$:

\[
\frac{1}{2} \pi_1^f \varphi^r < \left( \Pi^f - \pi_2^b \right) \left( \varphi^r + \frac{1}{2} \lambda \pi_1^f \varphi^{r-1} \right).
\]
where the inequality follows from the assumption that (4) should hold for \( s - 2 \) and for \( t = 1, 2, \ldots, \tau \). If we do the same exercise for \( s \) and \( s - 1 \), then we obtain

\[
\frac{p}{2} - \Delta_1 V_\tau(s) = \frac{p}{2} - \sum_{j=0}^{\tau-2} \left( 1 - \lambda \pi_2^b \right)^j \lambda \pi_2^b \Delta_1 V_{\tau-1-j}(s-2)
\]

\[
= \frac{p}{2} \left( 1 - \lambda \pi_2^b \right)^{\tau-1} + \sum_{j=0}^{\tau-2} \left( 1 - \lambda \pi_2^b \right)^j \lambda \pi_2^b \left( \frac{p}{2} - \Delta_1 V_{\tau-1-j}(s-2) \right).
\]

Consequently, we see that

\[
p - \Delta_2 V_\tau(s) > p \left( 1 - \frac{\delta}{2} \right) \left( 1 - \lambda \pi_2^b \right)^{\tau-1} + \delta \left( \frac{p}{2} - \Delta_1 V_\tau(s) \right).
\]

The assumption that \( \delta < 2 \) implies (4) holds for \( t = \tau + 1 \) and at the inventory level \( s \). Thus strategy \( b \) is optimal in that case.

Suppose that the optimality of strategy \( b \) is true only at \( s - 1 \) and \( s - 2 \) for \( t = 1, 2, \ldots, \tau \). We can inductively show that strategy \( b \) is also optimal at \( s \) for \( t = 1, 2, \ldots, \tau + 1 \). Indeed, the fact that strategy \( b \) is optimal at \( s \) for \( t = 1 \) and the above arguments yield that the retailer chooses \( b \) at \( s \) for \( t = 2 \). We can do backward induction until we arrive at \( t = \tau + 1 \) for the level \( s \).

**Part 2**

As in Part 1, the assumption on \( \Lambda^b \) and \( \Lambda^f \) is necessary and sufficient for the optimality of strategy \( f \) at \( t = 1 \) for any inventory level greater than 1. It also implies that \( \delta > 2 \). We first consider the case \( s \geq 4 \). Suppose that strategy \( f \) is optimal for \( t = 1, 2, \ldots, \tau \) at the inventory levels \( s - 2, s - 1, \) and \( s \). Then, the optimality equation reads

\[
V_t(s) = \lambda \pi_1^f \left( \frac{p}{2} + V_{t-1}(s-1) \right) + \lambda \pi_2^f (p + V_{t-1}(s-2)) + (1 - \lambda \pi_1^f - \lambda \pi_2^f) V_{t-1}(s)
\]

for \( t = 1, 2, \ldots, \tau \). With \( \varphi = 1 - \lambda \Pi^f \), we get

\[
V_t(s) - p = \varphi (V_{t-1}(s) - p) + \lambda \pi_1^f \left( V_{t-1}(s-1) - \frac{p}{2} \right) + \lambda \pi_2^f V_{t-1}(s-2).
\]

It is then not difficult to obtain

\[
V_t(s) = p \left( 1 - \varphi^t \right) + \sum_{j=0}^{t-1} \lambda \pi_1^f \varphi^j \left( V_{t-1-j}(s-1) - \frac{p}{2} \right) + \sum_{j=0}^{t-2} \lambda \pi_2^f \varphi^j V_{t-1-j}(s-2) \tag{5}
\]

for \( t = 1, 2, \ldots, \tau \). The same formula holds at the inventory levels \( s - 1 \) and \( s - 2 \) from which we get the expressions for \( \Delta_1 V_t(s) \) in terms of \( \Delta_1 V_{t-1-j}(s-1) \) and \( \Delta_1 V_{t-1-j}(s-2) \). In a straightforward manner, the next equalities follow:

\[
p - \Delta_2 V_\tau(s) = \sum_{j=0}^{\tau-2} \left\{ \lambda \pi_1^f \varphi^j (p - \Delta_2 V_{\tau-1-j}(s-1)) + \lambda \pi_2^f \varphi^j (p - \Delta_2 V_{\tau-1-j}(s-2)) \right\}
\]

\[
+ p - \frac{p}{2} \sum_{j=0}^{\tau-2} \varphi^j \lambda \Pi^f,
\]

27
\[\delta \left(\frac{p}{2} - \Delta_1 V_\tau(s)\right) = \sum_{j=0}^{\tau-2} \left\{ \lambda \pi_1^f \varphi^j \delta \left(\frac{p}{2} - \Delta_1 V_{\tau-1-j}(s-1)\right) + \lambda \pi_2^f \varphi^j \delta \left(\frac{p}{2} - \Delta_2 V_{\tau-1-j}(s-2)\right) \right\} + \delta \frac{p}{2} - \delta \frac{p}{2} \sum_{j=0}^{\tau-2} \varphi^j \lambda I^f.\]

The optimality of strategy \(f\) for inventory levels \(s - 1\) and \(s - 2\) and for \(t = 1, 2, \ldots, \tau\) implies that the inequality (4) is reversed for \(x = s - 1, s - 2\) and \(t = 1, 2, \ldots, \tau\). This observation with \(\delta > 2\) allows us to conclude that
\[p - \Delta_2 V_{\tau}(s) < \delta \left(\frac{p}{2} - \Delta_1 V_\tau(s)\right).\]

Therefore, it is optimal for the retailer to choose strategy \(f\) at the level \(s\) for \(t = \tau + 1\).

Now suppose that strategy \(f\) is optimal at \(s - 1\) and \(s - 2\) for \(t = 1, 2, \ldots, \tau\). Then, a similar argument as in Part 1 is applicable and we can confirm the optimality of \(f\) for \(t = 1, 2, \ldots, \tau + 1\). Since we proved the optimality of \(f\) for every \(t\) at \(s = 2\) in Lemma 2, it is enough to show the same is true at \(s = 3\). The desired result then follows from induction.

Let us fix \(s = 3\) and suppose that policy \(f\) is optimal up to \(t\). To simplify notation, we write \(x\) and \(y\) for \(\lambda \pi_1^f\) and \(\lambda \pi_2^f\), respectively. The equation (5) implies
\[V_\tau(3) = p(1 - \varphi^t) - \frac{p}{2} \frac{x}{x + y} (1 - \varphi^t) + \frac{t}{2} \sum_{j=0}^{t-2} \varphi^j \left\{ x V_{\tau-1-j}(2) + y V_{\tau-1-j}(1) \right\} = p(1 - \varphi^t) \frac{x + 2y}{2(x + y)} + p \left( x + \frac{y}{2} \right) \left( \frac{1 - \varphi^{t-1}}{1 - \varphi} - (t-1) \varphi^{t-1} \right) - \frac{p}{2} \left( x + \frac{y}{2} \right) \frac{2(x + 2y)}{2(x + y)} \varphi - \frac{p}{2} \left( x + \frac{y}{2} \right) \left( x + \frac{y}{2} \right) (t-1) \varphi^2 + \frac{x^2(t-1)}{4} G(t; x, y),\]
where we used \(V_\tau(1) = p(1 - \varphi^1)\), \(V_\tau(2) = p(1 - \varphi^t) - (p/2)x t \varphi^{t-1}\) and introduced an auxiliary function
\[G(t; x, y) = \frac{x + 2y}{2(x + y)} \varphi^2 + \frac{2x + y}{2(x + y)} \varphi + \left( x + \frac{y}{2} \right) (t-1) \varphi + \frac{x^2(t-1)}{4}.\]

One can easily verify that formula is correct for \(t = 1\) as well. To prove the optimality of policy \(f\) at \(t + 1\), we show the inequality in (4) is reserved, which is equivalent to
\[G(t; x, y) - \frac{1}{2} \varphi^2 < \delta \left( G(t; x, y) - \varphi^2 - \frac{1}{2} x t \varphi \right).\]

For \(t = 1\), the left side is \((1 - y/2) \varphi\) while the right side is \(\delta \varphi/2\). Since \(\delta > 2\), it is done. On the other hand, we compare the derivatives of both sides with respect to \(t\):
\[\left( x + \frac{y}{2} \right) \varphi + x^2 \left( \frac{t}{2} - \frac{1}{4} \right) < \delta \left\{ \left( \frac{x}{2} + \frac{y}{2} \right) \varphi + x^2 \left( \frac{t}{2} - \frac{1}{4} \right) \right\},\]
where the inequality holds again due to \(\delta > 2\). Therefore, the condition for the optimality of \(f\) holds for \(t + 1\). By induction, \(t\) can be arbitrarily large. The proof is complete.