A Recursive Method for Static Replication of Autocallable Structured Products

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Abstract

This paper discusses the problem of a valuation and risk management of structured products, which have been popular in recent financial markets. We propose a recursive method based on static replication for a variety of structured products, and, in particular, focus on products with autocallable and barrier features under a general Markovian diffusion with killing. The core idea of the proposed algorithm is to recursively utilize strike-spread approach and calender-spread approach in the literature. To increase computational and practical feasibilities, we devise discrete static hedges and their convergence is analyzed. Numerical experiments are conducted to confirm the effectiveness of our proposal and to show its highly accurate pricing and hedging performances.

KEYWORDS: options replication; derivative pricing; autocallable products; reverse convertibles;

1 Introduction

In this paper, we study static options replication for complex structured products. In the universe of structured products, we focus on products with autocallable features and/or barrier features. This type of products has been drawing an increasing amount of attention from academics as well as from practitioners in recent years. The size of the US market for autocallable products is documented in Deng et al. [2011, 2014]. According to the former, new issuances during the period of 2007–2010 amount to more than 40 billion dollars in the United States. On the other hand, Albuquerque et al. [2015] report that the total issue size was about 9.6 billion dollars during the period of mid-2009 to

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mid-2013 based on their search of SEC’s EDGAR database.\textsuperscript{1} Our motivation in this paper comes from the popularity of autocallable products in the financial market of South Korea. Based on the Bank of Korea [2013], the outstanding balance in mid-2012 for autocallable products, in particular reverse convertibles, was approximately 25 billion dollars.\textsuperscript{2}

Structured products could widely vary in their details. However, the main feature of autocallable products is the existence of trigger conditions. If the underlying asset meets a trigger condition, say up-crossing a predetermined strike at a predetermined date (call date), then the holder receives a reward and the contract terminates. An autocallable product may have fixed coupons as long as the contract is alive. It is also possible that the payoff at the maturity of a structured product is linked to the lowest value of the underlying asset, say down-crossing a pre-specified barrier level during the lifetime of the security. Then, it is called reverse convertibles. In conjunction with autocall features, it goes by autocallable (barrier) reverse convertibles.

With the low interest rate trend since early 2000, such structured products that offer attractive interest rates to investors have dramatically grown in financial markets. Major brokerage firms sell such products under different names. Despite such popularity and influence on financial markets, the risk management of autocallable barrier reverse convertibles remains a critical issue. Indeed, some financial institutions suffered severe losses from the failure of ELS risk management, e.g. Choi [2016]. Financial Times reported that Korea’s oversized ELS market is linked to worldwide financial networks, and was wary of potential risks [Hughes, 2016].

The complexity of their payoff structure hinders one from having a successful risk managing plan other than dynamic hedging. For one instance, let us consider a typical autocallable reverse convertible. When the underlying asset down-crosses a barrier, only the final payoff is affected not intermediate autocall features. Hence, embedded is a highly exotic barrier option. For another instance, in a highly volatile market, hedging a trigger event could be a difficult task to maneuver because the payoffs at call dates are discontinuous. There also have been some controversies between issuers and holders; see Lee [2017] for one settled case recently. Hence, a successful risk control is in the interests of firms and regulators.

Our proposal in this paper combines two existing strands of literatures in order to construct static hedging portfolios of autocallable barrier reverse convertibles. They are the so called calendar-spread approach and strike-spread approach. We review those ideas in Section 3. From the viewpoint of the product seller, the remaining task is to re-balance a hedging portfolio at a call date or at

\textsuperscript{1}These numbers do not contain reverse convertibles, which are also popular variants of autocalls.
\textsuperscript{2}To be precise, reverse convertibles constitute about 70\% of the so called equity-linked securities (ELS) in Korea. The outstanding balance of this ELS market was 35.9 billion dollars in mid-2012.
the barrier crossing once an initial static portfolio is constructed. At this re-balancing point, no additional cost would arise as long as transaction costs are negligible.

Guillaume [2015] provide a list of various callable products. The author also addresses the valuation problem under the Black-Scholes model. For more general structured products, the reader is referred to Deng et al. [2014] where four numerical valuation schemes are explained. Those methods are based on simulation, partial differential equations, decomposition of payoff structures, and numerical integration. Our proposed method can also be understood as yet another valuation scheme. Although we specifically deal with autocallable barrier reverse convertibles, ours can be applied to autocallable products summarized in Guillaume [2015].

The distinctive features of ours are first, its flexibility: the recursive method in this paper is applicable to combinations of autocalls, knock-in barriers, early redemption rewards, regular coupons and so on. Second, static hedging strategies are obtained as well as prices of target products. As pointed out by Derman and Taleb [2005], dynamic hedges may fail to generate satisfactory outcomes. Static replication offers important alternatives for hedgers to evaluate and utilize for successful risk control. Last but not least, while the literature is restricted to the classical Black-Scholes model due to the complex features of structured products, this paper considers general Markovian diffusion with killing as an underlying asset dynamics. For numerical illustrations, we consider the JDCEV model proposed by Carr and Linetsky [2006], which has attracted a lot of attention thanks to its ability to combine credit risk and market risk.

The paper is organized as follows: Section 2 presents a brief description of a modeling framework, basic options and structured products with callable and/or barrier features. Section 3 reviews two different static hedging strategies in the literature. Section 4 develops an algorithm to statically replicate the target option in a backward recursive manner. In Section 5, we propose discrete static hedges to enhance practical and computational feasibility. Section 6 provides numerical results, and Section 7 concludes.

2 Preliminaries

2.1 Modeling framework

We consider the modeling approach of Carr and Linetsky [2006] to construct a unified framework for the valuation of corporate liabilities, credit derivatives and equity derivatives. It is assumed that the market is frictionless and arbitrage-free, and that there exists an equivalent martingale measure \( \mathbb{Q} \). Under this pricing measure \( \mathbb{Q} \), the underlying asset price \( S_t \) is given as a one-dimensional Markovian
diffusion process with a possible jump-to-default. Due to the Markovian assumption, instantaneous volatility and default intensity are given as functions of \( S_t \) and \( t \).

Mathematically, we write the dynamics of \( S_t \) as the following stochastic differential equation:

\[
\frac{dS_t}{S_t} = [r - q + \lambda(S_t, t)]dt + \sigma(S_t, t)dW_t \tag{1}
\]

where we have \( S_0 > 0 \), the risk free interest rate \( r \geq 0 \), the continuous dividend yield \( q \geq 0 \), instantaneous volatility function \( \sigma(S_t, t) \), default intensity function \( \lambda(S_t, t) \). Lastly, \( W_t \) is a standard Brownian motion generating the filtration \( \{F_t\} \). Default can occur either at the first hitting time of zero for the diffusion \( S_t \), denoted by \( T_0 := \inf\{t > 0 : S_t = 0\} \), or at a jump-to-default time \( \tilde{\zeta} \).

The latter is defined as

\[
\tilde{\zeta} := \inf\left\{t > 0 : \int_0^t \lambda(S_u, u)du \geq \mathcal{E}\right\}
\]

where \( \mathcal{E} \) is an exponential random variable with unit mean independent of \( W \). Then, the default time \( \zeta \) is written as \( \zeta = T_0 \land \tilde{\zeta} \). After default time \( \zeta \), the asset price process is sent to a cemetery state defined as zero. The associated default indicator process \( 1_{\{t > \zeta\}} \) gives us the filtration \( \{D_t\} \). We, then, shall work with the enlarged filtration \( G_t = F_t \lor D_t \).

We can further specify \( \sigma \) and \( \lambda \) as in Carr and Linetsky [2006] although we do this only for numerical experiments. The so called JDCEV model proposed by the authors specifies the instantaneous volatility function as a power function of the stock price to capture the leverage effect and the implied volatility skew:

\[
\sigma(S_t, t) = a_t S_t^\beta
\]

where \( \beta < 0 \) is the volatility elasticity parameter and \( a_t > 0 \) is the time-dependent volatility scale parameter. The default intensity function is assumed to be an affine function of the instantaneous stock variance to accommodate the positive relationship between credit default swap spreads and equity volatilities:

\[
\lambda(S_t, t) = b_t + c \sigma(S_t, t)^2
\]

where \( c \geq 0 \) and \( b_t \geq 0 \). Note that the defaultable asset price is almost surely killed by a jump to default before the process \( S \) hits zero under this specification, yielding \( \zeta = \tilde{\zeta} \) almost surely.

### 2.2 European options

This subsection briefly presents notation and valuation for European contracts used in this paper. In the reduced form modeling literature for credit risk, a European option possibly has a fixed recovery upon default at maturity. For instance, while a European call option with strike \( K \), maturity \( T \) and payoff \((S_T - K)^+\) has no recovery upon default, the corresponding European put option with
payoff \((K - S_T)^+\) has two components: the put option with zero recovery \((K - S_T)^+1_{\{\zeta > T\}}\) and a recovery payment at maturity \(R1_{\{\zeta \leq T\}}\) if default occurs for some recovery payment \(R\).

Along this line, the value of a European claim with asset price \(x\), maturity \(T\) and payoff \(\varphi\) at time \(t\) is given by

\[
v_t(x; T, \varphi) = E\left[e^{-r(T-t)}\varphi(S_T)1_{\{\zeta > T\}} \mid \mathcal{G}_t\right] + E\left[e^{-r(T-t)}R1_{\{\zeta \leq T\}} \mid \mathcal{G}_t\right].
\]

Our statically replicating portfolios consist of vanilla options and binary options. Only for the simplicity of exposition, we assume zero recoveries throughout this paper.\(^3\) This allows us to write the values of our main European claims as follows:

\[
P_{t}^{\text{eur}}(x; T, K) = E\left[e^{-r(T-t)}(K - S_T)^+1_{\{\zeta > T\}} \mid \mathcal{G}_t\right],
\]

\[
C_{t}^{\text{eur}}(x; T, K) = E\left[e^{-r(T-t)}(S_T - K)^+1_{\{\zeta > T\}} \mid \mathcal{G}_t\right],
\]

\[
P_{t}^{\text{bin}}(x; T, K) = E\left[e^{-r(T-t)}1_{\{S_T < K, \zeta > T\}} \mid \mathcal{G}_t\right],
\]

\[
C_{t}^{\text{bin}}(x; T, K) = E\left[e^{-r(T-t)}1_{\{S_T > K, \zeta > T\}} \mid \mathcal{G}_t\right].
\]

They are respectively European put/call and binary put/call. Lastly, we introduce a defaultable zero-coupon bond with unit face value and no recovery:

\[
B_{t}(x; T) = E\left[e^{-r(T-t)}1_{\{\zeta > T\}} \mid \mathcal{G}_t\right].
\]

We refer the reader to Carr and Linetsky [2006] and Dias et al. [2014] for analytic formulas for the above basic options and the defaultable zero-coupon bond.

### 2.3 Description of autocallable products

This paper focuses on popular autocallable products with barrier features. In our numerical tests, we have two specifications: autocallable barrier reverse convertible notes (ABRCN) and stepdown knock-in equity linked securities (ELS). The former is sold and managed by European and US-based securities firms, and the latter constitutes a large portion of the ELS market of South Korea.

Although their payoff structures slightly vary, they can be generally characterized by the following features: autocall, knock-in barrier, regular coupon payment, early redemption reward. We emphasize that our approach is flexible enough to be applicable even when a target autocallable product is not a typical autocallable barrier reverse convertible. Here is a brief summary:

\(^3\)When the target contract has a recovery part upon default, we simply replicate the recovery part by including a credit default contract to our hedging portfolio. Thus, for ease of exposition, we only treat the zero recovery case in this paper.
Figure 1: Payoff structure of an autocallable barrier reverse convertible note with the initial price $S_0 = 100$. There are 3 early redemption opportunities with decreasing strikes $U_i$’s with fixed returns $r_i$. There could be separate coupons $c_i$. If the underlying ever touches the barrier $L$, then there could be a possible loss at the maturity (2-year) if the underlying is not greater than $U_4$.

1. autocall feature: on a semiannual basis, the contract terminates by paying its face value $F$ plus a fixed reward $r_i$ if the underlying up-crosses a strike $U_i$ at a call date $t_i$ for $i = 1, 2, \ldots, n - 1$.

2. coupon payments: the holder receives a fixed coupon $c_i$ at each $t_i$ as long as it is not called. If called at time $t_i$, then this $c_i$ is the last coupon payment.

3. barrier feature: the contract is knocked-in if the underlying down-crosses a barrier $L$. If this happens with no default during the life of the option, then the final payoff at the maturity $T = t_n$ (if not called) is $(F + r_n)1_{\{S_T \geq U_n\}} + S_T1_{\{S_T < U_n\}}$ plus the coupon $c_n$. If neither knocked-in nor called, then the contract terminates with the total return $F + r_n$ plus the coupon.

Figure 1 illustrates the payoff structure of our target product. This instance matures in two years. Call dates are 6, 12, and 18 months from the initial date. And at each call date, it is observed whether the underlying is greater than or equal to $U_i$. If that happens, the contract expires with reward $r_i$ (plus a possible coupon $c_i$). If it does not expire until the maturity, then a knock-in event is considered to determine the final payoff. If the barrier $L$ has never been breached, then the holder receives the reward $r_4$; otherwise, the holder loses some of her investment when the underlying does not exceed $U_4$. In the case of default, no cash will be exchanged between the issuer and the option holder thereafter.
The following formulas summarize the payoff structure of this example. At time $t_i$ for $i = 1, 2, \ldots, n - 1$, we have
\[
\prod_{j=1}^{i-1} 1\{s_{t_j} < u_j\} \left( (F + r_i) 1\{s_{t_i} \geq u_i\} + c_i \right) 1\{\zeta > t_i\}
\]
where the multiplier $\prod_{j=1}^{n-1} 1\{s_{t_j} < u_j\} = 1$ by convention. Lastly at the maturity $t_n$, the payoff structure is given by
\[
\prod_{j=1}^{n-1} 1\{s_{t_j} < u_j\} \left\{ \left( (F + r_n) 1\{s_T \geq u_n\} + S_T 1\{s_T < u_n\} \right) 1\{\tau \leq t_n\} + (F + r_n) 1\{\tau > t_n\} + c_n \right\} 1\{\zeta > t_n\}.
\]
Here $\tau$ is the first hitting time of $L$.

3 Methods of Static Replication

In our proposed algorithm for static replication, target callable products are decomposed into European contingent claims and exotic barrier options at each call date. Then, we apply the strike-spread approach to the European claims and the calendar-spread approach to the barrier options. This section reviews the two approaches for static options replication.

3.1 Calendar-spread approach

One of main difficulties in handling callable barrier reverse convertibles is that the knock-in event affects the final payoff at $T$ only when the contract has not been redeemed at a call date. This effectively makes the product a type of barrier options. However, unlike standard barrier options for which knock-in events convert barrier options into vanilla options, the knock-in event in our case turns the contract into a European claim with a highly complex payoff structure as we will see later.

For this reason, various exact hedging methods in the literature are not applicable. Instead, we apply the integral equations based approach proposed by Kim and Lim [2017]. In their paper, exotic barrier options with general payoffs at knock-in or knock-out boundaries are statically hedged with vanilla options of continuum of maturities. We record their main theorem which is tailored to our needs.

**Theorem 1 (Kim and Lim [2017])** Let $P_{t}^{d-1}(x;T,L,\varphi)$ be the time-$t$ value of a down-and-in barrier option with maturity $T$ and barrier $L$. Upon the knock-in event at time $t$, the option is to be converted into some European claim with value function $v_t(L;T,\varphi)$ for maturity $T$ and
payoff \( \varphi \). Assume that this value function along the barrier is continuous on \( t \in [0, T] \) and that 
\( v_T(L; T, \varphi) = \varphi(L) = 0 \). Then, a static hedge at time 0 is constructed as
\[
P_0^{d-i}(x; T, L, \varphi) = \int_0^T w(u) P_0^{eur}(x; T-u, L) du
\]
where \( w(\cdot) \) is a solution to the following Volterra equation of the first kind:
\[
\int_0^{T-t} w(u) P_t^{eur}(L; T-u, L) du = v_t(L; T, \varphi), \quad 0 \leq t \leq T.
\]

**Remark 1** Even though we assume \( \varphi(L) = 0 \) in the above theorem, it might be violated for some contracts such as reverse barrier options. In such a case, the replication is still possible by adding American binary options in our replicating portfolio. An American binary put with strike \( L \) gives one dollar to the option holder as soon as the underlying asset hits \( L \). Then, the static hedge is given by
\[
\int_0^T w(u) P_t^{eur}(x; T-u, L) du + v^* P_0^A(x; T, L)
\]
where \( v^* = \varphi(L) \). Here \( P_0^A \) is the time-0 value of the American binary put. The associated Volterra integral equation is also modified as
\[
\int_0^{T-t} w(u) P_t^{eur}(L; T-u, L) du = v_t(L; T, \varphi) - v^*, \quad 0 \leq t \leq T.
\]

The proof of this theorem relies on the principle of no arbitrage. The static hedging portfolio above replicates the payoffs of the target option for two scenarios: when the stock hits \( L \) before maturity and when the option expires worthless. The existence and uniqueness of such a solution, in particular that of \( w(\cdot) \), is relevant to the probabilistic properties of the asset dynamics \( S_t \). One can consult the literature on Volterra integral equations for sufficient conditions that need to be imposed on \( P_t^{eur} \) and \( v_t \). In this paper, we state one such condition in relation to Kim and Lim [2017] without proof. We will use this property to justify our hedging operations. We impose this condition as an assumption.

**Assumption 1** At-the-money implied volatility function is smooth and finite near maturity, and its time derivatives do not blow up near maturity.

Empirical studies on the behavior of implied volatilities support this assumption. It is also asserted by theoretical studies on model implied volatilities. For instance, see Dumas et al. [1998], Gatheral [2006], Gatheral et al. [2012]. If we have a specific model, then Assumption 1 can be
replaced by the properties of time sensitivities (known as *theta*) of hedging instruments: $v_t(L; T, \varphi)$ is twice continuously differentiable on $[0, T]$, and the theta of $P_{\text{eur}}$ satisfies the conditions in Theorems 2 and 3 of Kim and Lim [2017]. One can show that this holds as long as the asset dynamics is sufficiently nice, e.g., geometric Brownian motion or the jump-to-default extended CEV model.

On the other hand, if the model is time-homogeneous and if the Laplace transforms of $P_{\text{eur}}^0$ and $v_0$ exist, then we can find the Laplace transform of the hedge weight $w(\cdot)$. Here the Laplace transform $\hat{f}(\lambda)$ of a function $f(T)$ means the integral $\int_{0}^{\infty} e^{-\lambda T} f(T) dT$. Indeed, from the Volterra integral equation we obtain

$$\hat{v}(\lambda) = \int_{0}^{\infty} e^{-\lambda T} v_0(L; T, \varphi) dT = \int_{0}^{\infty} e^{-\lambda T} \int_{0}^{T} w(u) P_{\text{eur}}^0(L; T-u, L) du dT,$$

This leads us to $\hat{w}(\lambda) = \hat{v}(\lambda)/\hat{P}_{\text{eur}}^0(\lambda)$.

### 3.2 Strike-spread approach

In addition to a knock-in feature, another characteristic of our target autocallable products is the early redemption feature that makes their pricing and hedging complicated. In order to handle this, we limit our attention to each time interval between two consecutive call dates. Then, in each of those intervals, we construct a static hedging portfolio for an exotic European claim with a nonlinear and possibly discontinuous payoff. The next theorem prescribes how to build a static hedge for European claims with general payoffs.

**Theorem 2** Let $\varphi(x)$ be a twice continuously differentiable function on $(0, U)$ and $(U, \infty)$, respectively for a fixed $U > 0$. Further assume that $\varphi(U \pm)$ and $\varphi'(U \pm)$ exist in $\mathbb{R}$. Then, the time $t$-value of the European claim $v_t(x; T, \varphi)$ with maturity $T$ and the payoff $\varphi$ can be statically replicated by

$$v_t(x; T, \varphi) = \varphi(U-) P^\text{bin}_t(x; T, U) + \varphi(U+) C^\text{bin}_t(x; T, U)$$

$$- \varphi'(U-) P^\text{eur}_t(x; T, U) + \varphi'(U+) C^\text{eur}_t(x; T, U)$$

$$+ \int_{0}^{U} \varphi''(k) P^\text{eur}_t(x; T, k) dk + \int_{U}^{\infty} \varphi''(k) C^\text{eur}_t(x; T, k) dk.$$

Here, $P^\text{bin}_t(x; T, K)$ and $C^\text{bin}_t(x; T, K)$ are time-$t$ prices of binary put and call with maturity $T$ and strike $K$, respectively. Similarly, the prices of vanilla put and call are denoted by $P^\text{eur}_t$ and $C^\text{eur}_t$.

**Proof:** We begin by collecting binary puts, binary calls with maturity $T$ and strike $U$, and vanilla calls and puts with continuum of strikes. Their positions are to be determined by boundary
matching conditions: at time 0,

\[
\Pi_0(S_0) = w_1 P_{0}^{\text{bin}}(S_0; T, U) + w_2 C_{0}^{\text{bin}}(S_0; T, U) \\
- w_3 P_{0}^{\text{eur}}(S_0; T, U) + w_4 C_{0}^{\text{eur}}(S_0; T, U) \\
+ \int_{0}^{U} w_p(k) P_{0}^{\text{eur}}(S_0; T, k) dk + \int_{U}^{\infty} w_c(k) C_{0}^{\text{eur}}(S_0; T, k) dk,
\]

where \(w_i\)'s, \(w_p(\cdot)\), and \(w_c(\cdot)\) represent hedge weights of basic instruments.

Now suppose \(\zeta > T\) and \(0 < S_T = x < U\). Then, we get

\[
\Pi_T(x) = w_1 - w_3(U - x) + \int_{0}^{U} w_p(k)(k - x)^+ dk
\]

\[
= w_1 - w_3(U - x) + \int_{x}^{U} w_p(k)(k - x) dk.
\]

We want this to match \(\varphi(x)\) on \((0, U)\). The first differentiation yields

\[
\varphi'(x) = w_3 - \int_{x}^{U} w_p(k) dk.
\]

The second differentiation results in \(\varphi''(x) = w_p(x)\). On the other hand, we obtain \(w_1 = \varphi(U-)\) and \(w_3 = \varphi'(U-)\) by sending \(x\) to \(U\) from below.

In order to verify these specifications, we observe that, for \(0 < x < U\),

\[
\int_{0}^{U} \varphi''(k)(k - x)^+ dk = \int_{x}^{U} \varphi''(k)(k - x) dk
\]

\[
= \varphi'(k)(k - x) \bigg|_{x}^{U} - \int_{x}^{U} \varphi'(k) dk
\]

\[
= \varphi'(U -)(U - x) - \varphi(U -) + \varphi(x).
\]

By sending the first two terms on the right hand side to the left, we see that the portfolio value \(\Pi_T(S_T)\) matches \(\varphi(S_T)\) when \(S_T < U\).

Next, suppose \(S_T = x > U\). Then, the portfolio value at \(T\) is given by

\[
\Pi_T(x) = w_2 + w_4(x - U) + \int_{U}^{\infty} w_c(k)(x - k)^+ dk
\]

\[
= w_2 + w_4(x - U) + \int_{U}^{x} w_c(k)(x - k) dk.
\]

Straightforward differentiation from boundary matching gives us that \(\varphi'(x) = w_4 + \int_{U}^{x} w_c(k) dk\) and \(\varphi''(x) = w_c(x)\). Also, we easily observe that \(w_2 = \varphi(U+)\) and \(w_4 = \varphi'(U+)\). As a consequence, we arrive at the desired expression. The verification step for these specifications can be done similarly as above.

Here, we note that the portfolio \(\Pi_T(x)\) has value zero if the stock price hits zero (zero recovery assumption). The portfolio value \(\Pi_T(U)\) does not matter and our analysis is not affected because the event \(S_T = U\) has probability zero.
Remark 2 The number of discontinuity points can be arbitrarily extended. We note that this relation is a slight modification of the spanning relation of Carr and Madan [2001], allowing the payoff to have discontinuity points. Also, contrary to Carr and Madan [2001] who use Taylor expansion to prove the relation, we use integral equations which are more informative and helpful when approximating the result with finitely many options, and when analyzing convergence patterns of numerical schemes.

4 Main Algorithm

The algorithm is based on backward recursion. A statically replicating portfolio is constructed for each of the time intervals \([t_{i-1}, t_i]\) for \(i = 1, 2, \ldots, n\). At the call date \(t_i\), the target contract has some value, depending on whether the underlying \(S_{t_i}\) exceeds \(U_i\) or whether the knock-in event has occurred or not. If the barrier \(L\) has been hit before \(t_i\), then we can consider the contract as a European claim with a nonlinear payoff (the value function) at maturity \(t_i\). We utilize the strike-spread approach for this part. If the barrier has never been hit up to \(t_i\), then there is an embedded barrier option. We utilize the calendar-spread approach to statically handle this option. This procedure means that total \(n\) replicating portfolios are constructed. Rebalancing is required at each call date and when the underlying hits the barrier \(L\).

The applicability of the calendar-spread and strike-spread methods requires for us to assume that the asset model is nice enough to guarantee the smoothness of prices and thetas of certain European claims. Recall that \(v_t(x; T, \varphi)\) represents the time-\(t\) price of a European contingent claim with maturity \(T\) and payoff \(\varphi(x)\) provided \(S_T = x\).

Assumption 2 Let \(\varphi(\cdot)\) be a bounded and piecewise continuous payoff on \(\mathbb{R}_+\). The value function \(v_t(x; T, \varphi)\) is continuously differentiable in \(t\) and twice continuously differentiable in \(x\).

The above assumption holds if the asset price dynamics is sufficiently nice. The reader is referred to textbooks such as Kallenberg [2002] or Karatzas and Shreve [1991] for detailed conditions that ensure such regularity of a value function. In the option pricing context, Duffie [2001] is also a useful reference.

4.1 Initial construction

In this first step, we construct a static hedge at \(t_{n-1}\). Suppose that no default occurs by \(t_{n-1}\). This special case simplifies analysis because there is no early redemption opportunity between \(t_{n-1}\) and
$t_n$. Since the value of the contract at that time depends on whether or not the asset price has hit the knock-in barrier $L$ before $t_{n-1}$, we consider two cases separately. We introduce two symbols; $\Psi_1$ is the option value given that the barrier has not been breached whereas $\Psi_1^*$ is the value of the knocked-in option. Both symbols mean the option price conditional on $G_t$. Recall that we assume zero recoveries upon default.

4.1.1 First scenario

Suppose that the knock-in event has occurred before $t_{n-1}$ and that $S_{t_{n-1}} < U_{n-1}$. Then, the target autocallable barrier reverse convertible is simply a European claim with time-to-maturity $t_n - t_{n-1}$ and the strike $U_n$. More precisely, its payoff function is given by

$$\varphi_n(x) = \left( (F + r_n)1_{\{x \geq U_n\}} + x1_{\{x < U_n\}} + c_n \right) 1_{\{\zeta > t_n\}}.$$  

Its static hedging portfolio can be constructed as follows: for $t \in [t_{n-1}, t_n]$,

$$\Psi_t^*(x) = c_n B_t(x; t_n) + (F + r_n)C_t^{bin}(x; t_n, U_n) + U_nP_t^{bin}(x; t_n, U_n) - P_{t, t}^{eur}(x; t_n, U_n).$$  

In our notation, this can be expressed as $\Psi_t^*(x) = v_t(x; t_n, \varphi_n^*)$.

4.1.2 Second scenario

Now suppose that the knock-in event or early redemption has not happened until $t_{n-1}$. In this case, the option holder has missed all the early redemption events, but the risk of knock-in still exists. If the stock price hits $L$ before maturity, then the contract pays $\varphi_n^*$; otherwise, the payoff is

$$\varphi_n^0(x) = \left( (F + r_n)1_{\{x \geq L\}} + x1_{\{x < L\}} + c_n \right) 1_{\{\zeta > t_n\}}.$$  

This knock-in barrier option can be written as

$$\Psi_t^0(x) = v_t(x; t_n, \varphi_n^0) + P_{t, t}^{d-i}(x; t_n, L, \varphi_n^* - \varphi_n^0)$$  

for $t_{n-1} \leq t \leq \min\{t_n, \tau\}$, the earlier of the maturity or the hitting time of $L$. Here $P_{t, t}^{d-i}$ is a down-and-in barrier option that pays $\Psi_t^*(L) - v_t(L; t_n, \varphi_n^0) = v_t(L; t_n, \varphi_n^* - \varphi_n^0)$ at the knock-in barrier $L$. Or equivalently, $P_{t, t}^{d-i}$ is converted into a European contingent claim with maturity $t_n$ and payoff $\varphi_n^* - \varphi_n^0$ upon the knock-in event.

For the first term, we can make a static hedge as in the first scenario:

$$v_t(x; t_n, \varphi_n^0) = c_n B_t(x; t_n) + (F + r_n)C_t^{bin}(x; t_n, L) + LP_t^{bin}(x; t_n, L) - P_{t, t}^{eur}(x; t_n, L).$$
For the second term, we utilize the calendar-spread approach explained in Section 3.1. Vanilla puts are used for down-and-in options and we have
\[
P_{t_{n-i}}^d(x; t_n, L, \varphi_n^* - \varphi_n^o) = \int_{0}^{t_n-t} w(u) P_{t}^{eur}(x; t_n - u, L) du \quad \text{for } x \geq L.
\] (2)

The boundary matching condition is
\[
\int_{0}^{t_n-t} w(u) P_{t}^{eur}(L; t_n - u, L) du = v_t(L; t_n, \varphi_n^* - \varphi_n^o)
\]
for any time \( t \) on the interval \([t_{n-1}, t_n]\). Under the assumption that Laplace transforms of option prices are available and that the asset price dynamics is time-homogeneous, the hedge weights can be found explicitly. See the appendix.

**Remark 3** The payoffs \( \varphi_n^o \) and \( \varphi_n^* \) are not standardized. However, they usually consist of coupon payments when exceeding certain thresholds; otherwise, there are linear and negative returns. Hence, in applications, static hedges similar to the above can be found easily.

### 4.2 Backward recursion

Our strategy to build a static hedge is to perform a similar procedure as above in the backward manner. There is this initial construction which is valid for \([t_{n-1}, t_n]\), and \(n-1\) additional constructions, each of which is valid for each \([t_{i-1}, t_i]\), \(i = 1, 2, \ldots, n-1\). After these \(n\) steps, we shall arrive at a final statically replicating portfolio for \(\Psi_0^*(x)\) and the portfolio consists of basic European claims.

#### 4.2.1 First scenario

Suppose that the knock-in event has occurred before \(t_{i-1}\) and that \(S_{t_{i-1}} < U_{i-1}\). Here \(i\) is one of \(\{1, 2, \ldots, n-1\}\). Assume that we have the value of the target product \(\Psi_{t_i}^*(x)\) from the backward recursion. Since the option is already knocked-in, it can be thought of as a European claim with maturity \(t_i\) and the payoff
\[
\varphi_i^*(x) = \left( (F + r_i) \mathbf{1}_{\{x \geq U_i\}} + \Psi_{t_i}^*(x) \mathbf{1}_{\{x < U_i\}} + c_i \right) \mathbf{1}_{(\zeta > t_i)}.
\]

Recall the strike-spread approach from Section 3.2 so that we can find a static hedge for this type of European claims as follows:
\[
v_t(x; t_i, \varphi_i^*) = c_i B_t(x; t_i) + (F + r_i) C_t^{bin}(x; t_i, U_i) + \Psi_{t_i}^*(U_i -) P_{t}^{bin}(x; t_i, U_i)
\]
\[
- \frac{\partial \Psi_{t_i}^*}{\partial x}(U_i -) P_{t}^{eur}(x; t_i, U_i) + \int_{U_i}^{\infty} \frac{\partial^2 \Psi_{t_i}^*}{\partial x^2}(k) P_{t}^{eur}(x; t_i, k) dk
\]
where \(t_{i-1} \leq t \leq t_i\). Using our notation for \(v\), we have \(\Psi_i^*(x) = v_t(x; t_i, \varphi_i^*)\). This tells us how to construct a static hedge at \(t_{i-1}\). At \(t_i\), this portfolio is equal to \(F + r_i + c_i\) if \(S_{t_i} \geq U_i\); otherwise, it
is equal to $\Psi_t^*(x) + c_t$. Then, it is converted into the hedging portfolio constructed in the previous step for the interval $[t_i, t_{i+1}]$.

### 4.2.2 Second scenario

Now suppose that the knock-in event or early redemption has not happened until $t_i-1$. As in the previous subsection, consider two possibilities. If the stock price hits $L$ before $t_i$, then the contract’s payoff at $t_i$ is equivalent to $\varphi_i^*(x)$. Otherwise, it is same as the function value

$$\varphi_i^o(x) = (F + r_i)1_{x \geq U_i} + \Psi_t^o(x)1_{x < U_i} + c_t)1_{(\zeta > t_i)}.$$

Hence, if we confine ourselves to the interval $[t_{i-1}, t_i]$, this contract can be regarded as a knock-in barrier option whose value is decomposed into

$$\Psi_t^o(x) = v_t(x; t_i, \varphi_i^o) + P_{t_i}^{d-1}(x; t_i, L, \varphi_i^* - \varphi_i^o)$$

for $t_{i-1} \leq t \leq \min\{t_i, \tau\}$ where $\tau = \inf\{u > t_{i-1} : S_u = L\}$ and $P_{t_i}^{d-1}$ is a down-and-in barrier option with the payoff $\Psi_t^*(L) - v_t(L; t_i, \varphi_i^o) = v_t(L; t_i, \varphi_i^* - \varphi_i^o)$ at the knock-in barrier $L$.

For the first term, we apply the strike-spread approach in order to obtain

$$v_t(x; t_i, \varphi_i^o) = c_i B_t(x; t_i) + (F + r_i)C_t^{bin}(x; t_i, U_i) + \Psi_t^o(U_i-; P_{t_i}^{bin}(x; t_i, U_i))$$

$$- \frac{\partial \Psi_t^o(U_i-)}{\partial x} P_t^{eur}(x; t_i, U_i) + \int_0^{U_i} \frac{\partial^2 \Psi_t^o(U_i-)}{\partial x^2}(k) P_t^{eur}(x; t_i, k) dk.$$

For the second term, we apply the calendar-spread approach to derive

$$P_{t_i}^{d-1}(x; t_i, L, \varphi_i^* - \varphi_i^o) = \int_0^{t_i-t} w(u) P_t^{eur}(x; t_i - u, L) du \quad \text{for } x \geq L.$$

The boundary matching condition for this replication is $\int_0^{t_i-t} w(u) P_t^{eur}(L; t_i - u, L) du = v_t(L; t_i, \varphi_i^* - \varphi_i^o)$ for any time $t$ in the interval $[t_{i-1}, t_i]$. An analytic solution for $P_{t_i}^{d-1}$ can be derived through the Laplace transform if applicable.

### 5 Portfolio Construction with Finitely Many Options

#### 5.1 Discretization

**Strike-spread approach.** In our backward recursion, the payoff at each step is of the form $k1_{x \geq U} + \varphi(x)1_{x < U}$ multiplied by $1_{(\zeta > T)}$ where $\varphi$ is $C^2$ on $[0, U)$. The first term is easily replicated by binary calls. We apply Theorem 2 to the second term under the assumption that
\( \varphi(U-) \) and \( \varphi'(U-) \) exist. The resulting portfolio matches \( \hat{\varphi}(x) = \varphi(x)1_{\{x<U\}} \) for any possible realization of \( S_T \):

\[
v_0(x; T, \hat{\varphi}) = \varphi(U-)P_{0}^{\text{bin}}(x; T, U) - \varphi'(U-)P_{0}^{\text{eur}}(x; T, U) + \int_{0}^{U} \varphi''(k)P_{0}^{\text{eur}}(x; T, k)dk. \tag{3}
\]

Now suppose that there are \( m \) available put options with maturity \( T \) and strikes on the grid \( \{k_0 = 0 < k_1 < \cdots < k_m = U\} \). A natural approach for static replication would be to match the final payoff at these strikes. In other words, our new hedging portfolio is

\[
v_0^*(x; T, \hat{\varphi}) = \varphi(U-)P_{0}^{\text{bin}}(x; T, U) + \sum_{i=1}^{m} w_i P_{0}^{\text{eur}}(x; T, k_i)(k_i - k_{i-1}) \tag{4}
\]

and the boundary matching condition reads

\[
\varphi(U-) + \sum_{i=j+1}^{m} w_i (k_i - k_j)(k_i - k_{i-1}) = \varphi(k_j), \quad j = 0, 1, \ldots, m - 1. \tag{5}
\]

This system of linear equations is straightforward to solve. Regarding the quality of the approximation, we have the following result.

**Theorem 3** Consider a European claim with maturity \( T \) and payoff \( \hat{\varphi}(x) = \varphi(x)1_{\{x<U\}} \) where \( \varphi \) is sufficiently smooth on \((0, U)\) with finite left derivatives at \( U \). Assume that \( \Delta_i = O(m^{-1}) \) and \( \frac{\Delta_i}{\Delta_{i+1}} = 1 + O(m^{-1}) \) where \( \Delta_i = k_i - k_{i-1} \) for \( i = 1, \ldots, m \). If the approximating portfolio (4) satisfies (5), then we have \( \lim_{\Delta_i \to 0} v_t^*(x; T, \hat{\varphi}) = v_t(x; T, \hat{\varphi}) \) uniformly in \((t, x) \in [0, T] \times \mathbb{R}_+\).

**Proof:** See the appendix. \( \square \)

**Calendar-spread approach.** In our static hedging portfolio, down-and-in barrier options are used in order to handle the knock-in event of the lower barrier \( L \). Let us consider the barrier option \( P_{t}^{\text{d}-i}(x; T, L, \varphi) \) in Theorem 1 that becomes a European option with payoff \( \varphi \) once \( L \) is hit. Now suppose that there are put options with strike \( L \) and \( l \) different maturities, say \( \{\tau_0 = 0 < \tau_1 < \cdots < \tau_l = T\} \). Then, for a hedging portfolio

\[
P_{0}^{\text{d}-i}(x; T, L, \varphi) = \sum_{i=1}^{l} u_i P_{0}^{\text{eur}}(x; \tau_i, L) \Delta_i \tag{6}
\]

with \( \Delta_i = \tau_i - \tau_{i-1} \), the boundary matching condition reads

\[
\sum_{j=i+1}^{l} u_j P_{\tau_i}^{\text{eur}}(L; \tau_j, L) \Delta_j = v_{\tau_i}(L; T, \varphi), \quad i = 0, 1, \ldots, l - 1. \tag{7}
\]
Here, \( u_j \) corresponds to \( w(T - \tau_j) \) for the weight function \( w \) in Theorem 1. Indeed, the left side of (7) can be thought of as an approximation to \( \int_{\tau_i}^{T} w(T - u)P_{\tau_i}^{\text{eur}}(L; u, L)du \) by a simple rectangular rule, which is in turn equal to \( \int_{0}^{T-\tau_i} w(u)P_{\tau_i}^{\text{eur}}(L; T - u, L)du \). This integral appears in the target Volterra integral equation.

By inspecting (7), one can easily identify the solution to the system of linear equations. The established hedging portfolio is essentially equivalent to the replicating portfolio proposed by Derman et al. [1995] and many subsequent papers on calendar-spread approach. Unfortunately, the first kind Volterra equation is typically ill-posed. 4 For an ill-posed problem, an approximate solution obtained from a simple discretization may not converge to the true solution. (See Chapter 15 of Kress [2014] and Lamm [2000] for instance.) In other words, there is no general result to guarantee the convergence of \( u_j \) to \( w(T - \cdot) \). Nonetheless, the convergence of weights as well as good hedging performances have been numerically verified from our extensive experiments. We refer the reader to Section 6.1 for more information.

Albeit this lack of precise convergent analysis, there are two indirect ways of confirming that the performance of discrete versions like (6) can be close to perfect replication. The first and easier case is to set \( u_i = w(T - \tau_i) \) in (6) where \( w \) is the solution to the target Volterra equation. Then, the portfolio value \( P^d_{t-i}(L; T, L, \varphi) \) would not match \( v_t(L; T, \varphi) \) at \( \tau_i \)'s. Although this portfolio is likely to yield worse performance compared to the portfolio with boundary matching, we have a simple convergence result as follows: for each \( t \), there exists \( i \) such that \( \tau_{i-1} \leq t \leq \tau_i \) and we have

\[
\int_{t}^{T} w(T - u)P_{t}^{\text{eur}}(x; u, L)du = \int_{\tau_i}^{T} w(T - u)P_{\tau_i}^{\text{eur}}(x; u, L)du + O(\Delta_i) \\
\approx \sum_{j=i+1}^{l} w(T - \tau_j)P_{\tau_j}^{\text{eur}}(x; \tau_j, L)\Delta_j.
\]

In other words, we write \( \lim_{|\Delta| \to 0} P^d_{t-i}(x; T, L, \varphi) = P^d_{t-i}(x; T, L, \varphi) \). We note that this convergence is uniform in \((t, x) \in [0, T] \times [L, \infty)\).

A second case involves static options replication using binary puts instead of European puts. Kim and Lim [2017] prove that there exists a unique continuous \( w \) such that \( \int_{0}^{T} w(u)P_{0}^{\text{bin}}(x; T, L)du \) replicates the target barrier option in Theorem 1. We aim to show that the boundary matching condition (7) for this case indeed guarantees the convergence. For this, we need the following mild assumption on binary puts.

---

4Jacques Hadamard defined a well-posed problem that have three properties: 1. a solution exists, 2. the solution is unique, 3. the solution depends continuously on the initial condition. An ill-posed problem is one which does not satisfy the Hadamard criteria.
Assumption 3 Let $P_{0}^{\text{bin}}(x; T, L)$ be the time-$t$ price of a binary put with maturity $T$ and strike $L$. Then, $\lim_{t \to T} P_{t}^{\text{bin}}(L; T, L)$ converges to some non-zero constant as $t$ approaches $T$. Furthermore, its time derivative is weakly singular with parameter $\alpha \in (0, 1)$, i.e. $\Theta_{t}(L; T, L) = f(t, T)/(T - t)^{\alpha}$ for some continuous function $f$.

The limiting result and weak singularity of theta can be verified for well known financial models such as JDCEV model. See Kim and Lim [2017] for more details.

Theorem 4 Consider the approximating portfolio $P_{0}^{d-i}(x; T, L, \varphi) = \sum_{i=1}^{l} u_{i} P_{0}^{\text{bin}}(x; \tau_{i}, L)$ with boundary matching condition similar to (7). Then, under Assumptions 1–3 with $|\Delta| \to 0$, the approximating portfolio $P_{t}^{d-i}(x; T, L, \varphi)$ converges to $P_{t}^{d-i}(x; T, L, \varphi)$ uniformly in $(t, x) \in [0, T] \times [L, \infty)$ as long as $\Delta_{ij} = 1 + O(l^{-1})$ for all $i, j$.

Proof: See the appendix.

5.2 Discrete hedging with $\varepsilon$ error bound

The goal of this subsection is to show that one can construct a discrete hedging portfolio which approximates a given autocallable barrier reverse convertible product within a fixed $\varepsilon$ error. For this, we first consider a discrete portfolio $v_{t}^{*}$ in Theorem 3 and a discrete portfolio $P_{t}^{d-i}$ with weights $w(T - \tau_{i})$'s. Next, we observe that such approximations are actually uniform in $(t, x)$ due to the finiteness and continuity of $w$ and $P_{\text{eur}}$.

We have the following two lemmas.

Lemma 1 Suppose $\varphi$ and $\varphi'$ satisfy $\|\varphi - \varphi'\|_{\infty} \leq \varepsilon$. Then, $\sup_{(t,x)} |v_{t}(x; T, \varphi - \varphi')| \leq \varepsilon$.

Pricing via risk-neutral expectation yields this trivial result. Now let us consider two down-and-in barrier options with the same barrier $L$ such that the knock-in event turns them into European claims with payoffs $\varphi$ and $\varphi'$, respectively.

Lemma 2 Suppose $P_{t}^{d-i}(x; T, L, \varphi)$ and $P_{t}^{d-i}(x; T, L, \varphi')$ be down-and-in options with associated payoffs $\varphi$ and $\varphi'$. We further assume that default or knock-in has not occurred by the time $t$. If $\|\varphi - \varphi'\|_{\infty} \leq \varepsilon$, then $\sup_{(t,x)} |P_{t}^{d-i}(x; T, L, \varphi) - P_{t}^{d-i}(x; T, L, \varphi')| \leq \varepsilon$.

Proof: We define two portfolios $\Pi_{1}^{t}(x) = P_{t}^{d-i}(x; T, L, \varphi) - \varepsilon P_{t}^{A}(x; T, L)$ and $\Pi_{2}^{t} = P_{t}^{d-i}(x; T, L, \varphi) + \varepsilon P_{t}^{A}(x; T, L)$ where $P_{t}^{A}(x; T, L)$ is the time-$t$ price of an American binary put with maturity $T$ and
strike $L$ whose payoff is $1_{\{\tau \leq T, \zeta > \tau\}}$ at the hitting time $\tau = \inf\{s > t| S_s = L\}$. We will compare the values of $\Pi^1$ and $\Pi^2$ for three possibilities for each sample path.

Firstly suppose $\zeta < \min\{T, \tau\}$. Both $\Pi^1$ and $\Pi^2$ become worthless since we assume zero recovery for all option contracts. Secondly suppose $T < \min\{\zeta, \tau\}$. Then, $\Pi^1$ and $\Pi^2$ expire worthless without knock-in.

Lastly suppose $\tau < \min\{T, \zeta\}$. At time $\tau$, the two portfolios have values $v_\tau(L; T, \varphi) - \varepsilon$ and $v_\tau(L; T, \varphi) + \varepsilon$. From the assumption $\|\varphi - \varphi'\|_\infty \leq \varepsilon$ and Lemma 1, we have

$$\|v_\tau(L; T, \varphi) - v_\tau(L; T, \varphi')\|_\infty \leq \varepsilon$$

which in turn implies $\Pi^1(\tau) = v_\tau(L; T, \varphi) - \varepsilon \leq v_\tau(L; T, \varphi') \leq v_\tau(L; T, \varphi) + \varepsilon = \Pi^2(\tau)$. Therefore, one can apply no arbitrage principle to conclude $\Pi^1(x) \leq P^{d-i}(x; T, \varphi') \leq \Pi^2(x)$. By definition of $\Pi^1$ and $\Pi^2$, we see

$$\left| P^{d-i}(x; T, L, \varphi) - P^{d-i}(x; T, L, \varphi') \right| \leq \varepsilon P^A(x; T, L) \leq \varepsilon$$

for all $x$ and $t$. Note that the price of American binary put is less than 1 by the definition.

\[\blacksquare\]

**Theorem 5** Suppose that there are vanilla puts or binary puts with strike $L$ and sufficiently many maturities in $[0, T]$, and vanilla puts with maturity $T$ and sufficiently many strikes on $(0, U]$. Under Assumptions 1–3, there exists a portfolio $\Pi^0$ with finitely many options such that $|\Pi^0_0 - \Psi^0_0| \leq \varepsilon$ for an arbitrarily given $\varepsilon$.

**Proof:** Recall that a hedging portfolio for $\Psi^*_0(x)$ can be statically constructed with finitely many instruments for $t \in [t_{n-1}, t_n]$. For $\Psi^0_t = v_t(x; t_n, \varphi^0_n) + P^{d-i}(x; t_n, L, \varphi^*_n - \varphi^0_n)$, the first term $v_t$ is replicated with a small number of instruments. Regarding the barrier part $P^{d-i}_t$, we saw that there exists an approximate portfolio $P^{d-i*}$ with weights $w(T - \tau)$'s in Section 5.1 such that

$$\|P^{d-i}(x; t_n, L, \varphi^*_n - \varphi^0_n) - P^{d-i*}(x; t_n, L, \varphi^*_n - \varphi^0_n)\|_\infty \leq \varepsilon.$$

Here the sup norm is taken over $(t, x) \in [0, T] \times [L, \infty)$. A different approximate portfolio can be constructed based on Theorem 4 when binary puts are hedging instruments. As a result, we find an approximate discrete portfolio $\Pi^0_t(x; t_n) = v_t(x; t_n, \varphi^0_n) + P^{d-i*}_t(x; t_n, L, \varphi^*_n - \varphi^0_n)$ with $\varepsilon$ error bound for $t \in [t_{n-1}, t_n]$. For notational consistency, we define $\Pi^*_t(x; t_n) = v_t(x; t_n, \varphi^*_n)$.

Now, we consider the interval $[t_{n-2}, t_{n-1}]$. Recall we have the decomposition

$$\Psi^0_t = v_t(x; t_{n-1}, \varphi^0_{n-1}) + P^{d-i}_t(x; t_{n-1}, L, \varphi^*_{n-1} - \varphi^0_{n-1}). \quad (8)$$
Let us compare two payoff functions:

\[
\phi_{n-1}^0(x) = \left((F + r_{n-1})1_{\{x \geq U_{n-1}\}} + \psi_{tn-1}^0(x)1_{\{x < U_{n-1}\}} + c_{n-1}\right)1_{\{\zeta > t_{n-1}\}},
\]

\[
\phi_{n-1}^{o'}(x) = \left((F + r_{n-1})1_{\{x \geq U_{n-1}\}} + \Pi_{tn-1}^0(x; t_{n-1})1_{\{x < U_{n-1}\}} + c_{n-1}\right)1_{\{\zeta > t_{n-1}\}}.
\]

Lemma 1 implies the value functions of European claims with payoffs \(\phi_{n-1}^0\) and \(\phi_{n-1}^{o'}\) are different within \(\varepsilon\) bound. Then, as shown in Section 5.1, we can find a discrete portfolio \(v_t^* (x; t_{n-1}, \phi_{n-1}^{o'})\) that approximates \(v_t(x; t_{n-1}, \phi_{n-1}^{o'})\) within \(\varepsilon\) bound. Combining these two approximations, we construct an approximate hedge \(v_{t_{n-1}}^*\) to the first term in (8) within \(2\varepsilon\) bound.

Momentarily, we look at \(\psi_{tn-1}^0\) for \(t \in [t_{n-2}, t_{n-1}]\). We can similarly define \(\phi_{n-1}^{o'}(x)\) as above. This case is simpler because \(\psi_{tn-1}^0\) consists of finitely many instruments; hence no approximation is required. In backward recursion steps, approximate hedges shall be constructed though. Since it can be done in exactly the same way as above, we skip the details noting that an approximate hedge \(v_t^* (x; t_{n-1}, \phi_{n-1}^{o'})\) can be found to be within \(2\varepsilon\) bound from \(v_t(x; t_{n-1}, \phi_{n-1}^{o'})\) if \(\|\phi_{n-1}^{o'} - \phi_{n-1}^0\|_{\infty} \leq \varepsilon\). As a result, we have a discrete hedging portfolio \(\Pi_t^0(x; t_{n-1}) = v_t^* (x; t_{n-1}, \phi_{n-1}^{o'})\) for \(t \in [t_{n-2}, t_{n-1}]\).

Now let us turn our attention to an error bound for the barrier part \(P_t^{d-i}\). Lemma 2 tells us that there exists a barrier option \(P_t^{d-i}(x; t_{n-1}, L, \phi_{n-1}^{o'} - \phi_{n-1}^0)\) such that

\[
\left\|P_t^{d-i}(x; t_{n-1}, L, \phi_{n-1}^{o'} - \phi_{n-1}^0) - P_t^{d-i}(x; t_{n-1}, L, \phi_{n-1}^0 - \phi_{n-1}^0)\right\|_{\infty} \leq 2\varepsilon.
\]

The simple discretization of Section 5.1 would then yield an approximate hedge \(P_t^{d-i^*}\) with \(\varepsilon\) error bound from this intermediate barrier option, and in turn achieves \(3\varepsilon\) error from the target barrier option. More precisely, we get

\[
\left\|P_t^{d-i^*}(x; t_{n-1}, L, \phi_{n-1}^{o'} - \phi_{n-1}^0) - P_t^{d-i^*}(x; t_{n-1}, L, \phi_{n-1}^0 - \phi_{n-1}^0)\right\|_{\infty} \leq 3\varepsilon.
\]

The final construction \(\Pi_t^0(x; t_{n-1}) = v_t^* (x; t_{n-1}, \phi_{n-1}^{o'}) + P_t^{d-i^*}(x; t_{n-1}, L, \phi_{n-1}^{o'} - \phi_{n-1}^0)\) approximates \(\psi_t^0\) with the maximum \(5\varepsilon\) error for \(t \in [t_{n-2}, t_{n-1}]\). By resizing the target error, the error bound can be chosen to be \(\varepsilon\) at time \(t_{n-2}\). We repeat this procedure until we arrive at the first interval \([t_0, t_1]\) with suitable resizing of error bound. Then, we obtain an approximate discrete portfolio \(\Pi_0^0\) with \(\varepsilon\) error bound with respect to \(\psi_0^0\).

### 6 Numerical Results

Our proposal to use the backward recursive scheme applies to any general diffusions. To confirm the effectiveness of the proposed approach in pricing and hedging, we conduct an numerical experiment...
Table 1: Summary of product specification.

<table>
<thead>
<tr>
<th>Product</th>
<th>Maturity</th>
<th>$F$</th>
<th>$c_i$</th>
<th>$r_i$</th>
<th>$U_i$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELS</td>
<td>3 year</td>
<td>100</td>
<td>(0,0,0,0,0)</td>
<td>(2,4,6,8,10,12)</td>
<td>(95,95,95,90,90,90)</td>
<td>55</td>
</tr>
<tr>
<td>ABRCN</td>
<td>3 year</td>
<td>100</td>
<td>(4,4,4,4,4)</td>
<td>(0,0,0,0,0)</td>
<td>(100,100,100,100,100,100)</td>
<td>63</td>
</tr>
</tbody>
</table>

on pricing and hedging ABRCN and ELS under the jump-to-default extended CEV model. Specifically, we consider step-down ELS issued by Mirae Asset Daewoo in March, 2017, and ABRCN issued by JP Morgan in February, 2014. Their contracts’ specifications are summarized in Table 1. Autocall is evaluated every 6 months since the inception of a product.

6.1 Performance of the method of Derman et al. [1995]

Before presenting the pricing and hedging performance of our approach, we note that Theorem 5 is built on the approximation scheme in Theorem 3 and the simple discretization scheme of calendar-spread approach with true weights $w(T - \tau_i)$’s. Hence, it is worth investigating the performance differences between this approach and the method described in (6) and (7), called the DEK method.

For an illustration, we replicate a down-and-in barrier call option with barrier 55, strike 60, maturity 1 year and $\varphi = (S_T - K)^+$. It is assumed that the underlying stock follows the jump-to-default extended CEV model with parameters $a = 20.16$, $b = 0.02$, $c = 0.75$, $r = 3\%$, and $\beta = -1$. Thanks to the time homogeneity, we can obtain the true weights for calendar-spread approach via Laplace transform. The left panel of Figure 2 shows the weights from the DEK method and the true weights by solving the Volterra equation on the uniform time grid on $[0, 1]$ with six points. The right panel compares the performances of two approximation schemes in terms of matching the target option values along the boundary $L$, that is, $P_{t,i}^d(L; T, L, \varphi)$. One can visually confirm that (6) with (7) is better, which results in a discrete hedging portfolio with smaller replication errors.

Furthermore, we plot the convergence pattern of the weights $u_j$ computed from the DEK method. To construct the DEK hedging portfolio, 4 different time steps are used, that is, $l = 6, 24, 48, 96$ in (6) and (7). Figure 3 indicates that the weights $u_j$ apparently converge to the true weights obtained by the Laplace transform as the number of hedging instruments increases ($l$ gets larger). However, more evidences on properties of the weight function such as differentiability are required to rigorously prove the convergence of the DEK method when the hedging instrument is European put.
Figure 2: (Left) Approximate solutions from the two discretization methods. (Right) Values of hedging portfolios and a standard down-and-in barrier call with $L = 55$, $K = 60$, and $T = 1$. Parameter values for the JDCEV model are $a = 20.16$, $b = 0.02$, $c = 0.75$, $r = 0.03$ and $\beta = -1$.

6.2 Pricing

In this subsection, we intend to compute prices of ABRCN and ELS based on the recursive method with respect to various parameter configurations. In our numerical examples, different combinations of strikes and maturities, say (nb. of strikes, nb. of maturities) = {(4,6),(8,6),(20,6),(20,c)} where c represents a continuum of maturities, are used to see the effect of discretization techniques. For example, if it is (4,6), then we use six equidistant maturities to replicate $P^{d-i}$ and four equidistant strikes to replicate $v_t$ for each interval $[t_{i-1}, t_i]$.

The numerical results are given in Table 2 where we also provide Monte Carlo simulation results with the standard error as a benchmark. To obtain the Monte Carlo value, we employ 500,000 sample paths with 3,000 time steps, which are generated by a discretization method shown in Giesecke and Smelov [2013]. The standard error is estimated as the sample standard deviation divided by the square root of the number of sample paths. Model parameters $b$ and $c$ gauge the intensity of a default event and $\beta$ controls the leverage effect of the underlying asset. This $\beta$ also determines the instantaneous volatility level together with parameter $a$.

In Table 2, we observe that four prices computed from different discrete grids for each row are quite similar, implying that an approximate hedging portfolio with a small number of strikes and maturities could replicate ELS and ABRCN successfully. In particular, when we use 20 strikes to construct hedging portfolios, their prices stay within 1 standard errors of the associated Monte Carlo value.
Figure 3: The convergence of the weights computed from the DEK method. The solid line represents the true weight computed from Laplace transform and the dotted line represents the weight of the DEK method with European put options. 4 different time grids are used to implement the DEK method, namely \(l = 6, 24, 48, 96\) in (6) and (7).

Carlo values. This provides a strong evidence that the recursive method achieves almost exact prices with practically available strikes and maturities.

6.3 Hedging

The performance of statically replicating portfolios is examined in this subsection. We conduct simulation experiments by generating 500 scenario paths under the JDCEV model with \(a = 20.16, b = 0.02, c = 0.75\) and \(\beta = -1.5\). For the simplicity’s sake, we generate scenario data under the risk neutral measure. In each simulation, we imagine a situation where an ELS, specified in Table 1, is

\(^5\)To economize on space, we report results with one parameter setting only, but results with different configurations are available upon request.
Table 2: Pricing results of ELS and ABRCN. Note that $S_0 = 100$, $a = 0.2016S_0^\beta$, $r = 0.03$ and $T = 3$. Standard errors for Monte Carlo simulation are given in brackets. In the “recursive” column, $c$ represents a continuum of maturities.

<table>
<thead>
<tr>
<th>product</th>
<th>$b$</th>
<th>$c$</th>
<th>$\beta$</th>
<th>Monte Carlo (se)</th>
<th>recursive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(4,6)</td>
</tr>
<tr>
<td>ELS</td>
<td>0.02</td>
<td>0.75</td>
<td>-1</td>
<td>94.72 (0.032)</td>
<td>94.50</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>0.75</td>
<td>-2</td>
<td>94.37 (0.034)</td>
<td>94.25</td>
</tr>
<tr>
<td>ABRCN</td>
<td>0.02</td>
<td>0.75</td>
<td>-1</td>
<td>97.43 (0.034)</td>
<td>97.16</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>0.75</td>
<td>-2</td>
<td>96.66 (0.036)</td>
<td>96.46</td>
</tr>
</tbody>
</table>

Table 3: Statistics of ELS payoff. The number in each cell represents the number of corresponding scenarios.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>nb. of knock-ins</th>
<th>nb. of defaults</th>
<th>expiry date (year)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>19</td>
<td>370</td>
</tr>
<tr>
<td>70</td>
<td>141</td>
<td>105</td>
<td>33</td>
</tr>
</tbody>
</table>

Table 3 reports the summary statistics of ELS payoff for 500 simulated paths with the initial asset price 100 and 70. When the initial price is 100, only 2% of paths cross the knock-in barrier. ‘expiry date’ column shows that the ELS expires at the first autocall date for 74% among all scenarios considered. This situation works favorably for the hedger because hedging errors from discretization do not accumulate beyond the first call date under those scenarios. By contrast, it is observed that 28.2% of samples breach the knock-in barrier if the initial price is 70. Furthermore, the frequency of the first autocall decreases considerably. Interestingly though, under 30.6% of scenarios the ELS is alive until maturity without autocalls nor default.

We perform static hedging for the above simulated paths under the assumption that the hedger has exact and full information about the parameters of the underlying process and zero friction. In other words, this experiment does not take account of model risk, parameter estimation errors nor transaction costs. Similarly to the pricing part, hedging portfolios are made of two different combinations: (nb. of strikes, nb. of maturities) = {(4,6),(8,6)}. Hedging operation could be terminated prior to the maturity when autocall or default happens. There are at most six portfolio
re-balancing opportunities (at 5 call dates and one possible knock-in event date) and a self-financing trade is kept during the hedging period.

There is a subtlety in defining hedging errors. Since our target is to create, maintain, and re-balance discrete hedging portfolios, we aim to possess $\Pi_t^\circ(x; t_i)$ or $\Pi_t^\bullet(x; t_i)$ in the proof of Theorem 5 for $i = 1, 2, \ldots, n$ before maturity. At the termination of the contract, the difference between the payoffs of the ELS and a hedging portfolio is hedging error. At re-balancing, the hedger might need to borrow cash to keep the target hedge. For instance, suppose the barrier has not been crossed up to $t_i$ and the hedger wants to construct the hedge $\Pi_t^\circ(x; t_{i+1})$. Then, cash transactions are required at $t_i$ for the amount of $\Pi_t^\circ(x; t_{i+1}) - \Pi_t^\circ(x; t_i)$. This is also considered as hedging error in our experiments. Consequently, hedging error is defined as the sum of their discounted values.

The sources of hedging errors are briefly summarized. We note that there are two different kinds of discretization errors. The first one is the error stemmed from the application of discrete calendar-spread approach and discrete strike-spread approach at each time periods $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. For a given payoff $\varphi^\circ_i$ and $\varphi^\bullet_i$, we constructed approximate portfolios $v_{t_{i-1}}$ for $v_{t_i}$ and $P^d_{t_{i-1}}$ for $P^d_{t_i}$ in Section 5.2. The second one is the error from the backward recursive steps. The tilted payoffs, $\varphi^\circ_i$ and $\varphi^\bullet_i$, to the true payoffs, $\varphi^\circ_i$ and $\varphi^\bullet_i$, involve these errors. For instance, there is a value difference between $v_{t_{i-1}}(x; t_i, \varphi^\circ_i)$ and $v_{t_{i-1}}(x; t_i, \varphi^\circ_i)$. Theorem 5 proves that these discrepancies are properly controlled by the suggested discrete hedging schemes.

Table 4 summarizes several statistics of hedging errors: mean, standard deviation, min, max, value-at-risk or simply VaR. As noted previously, if the initial asset price is 100, then hedging errors are expected to be small because the ELS expires at the first call date for 74% of simulated scenarios. Indeed, for instance, hedging errors of a discrete static hedge with 4 strikes and 6 maturities are located within $[-3.9571\%, 1.1568\%]$ when $S_0$ is 100. Furthermore, the hedging performance gets better if there are eight strikes instead of four. If the initial price is 70, the performance of static strategies gets slightly worse because portfolio re-balancing is required multiple times due to the increased knock-in risk and smaller probabilities of autocalls. However, the results still seem satisfactory; e.g., less than 1% error on average with a small number of vanilla options.

The interesting feature is that most of hedging errors are negative. This fact is clearly visualized by looking at the replicating patterns for discrete strike-spread and discrete calendar-spread portfolios. The left panel in Figure 4 depicts how a strike-spread portfolio with 8 strikes works. The solid line represents target ELS values at 0.5 year and the dotted line represents static hedging portfolio values with respect to the asset price ranging from 55 to 95. Their mismatch values are shown in the right panel in Figure 4. It indicates that the discrete strike-spread approach tends to be
Table 4: Statistics of hedging errors. (str, mat) means the nb. of strikes and the nb. of maturities that are used to construct a static hedge. std means the standard deviation. VaR_\alpha is the \alpha-level quantile of the hedging error distribution.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>(str, mat)</th>
<th>mean (%)</th>
<th>std (%)</th>
<th>min (%)</th>
<th>max (%)</th>
<th>VaR_{0.01} (%)</th>
<th>VaR_{0.99} (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>(4,6)</td>
<td>-0.1942</td>
<td>0.5567</td>
<td>-3.9571</td>
<td>1.1568</td>
<td>-3.1842</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(8,6)</td>
<td>-0.0627</td>
<td>0.2084</td>
<td>-1.6744</td>
<td>1.9158</td>
<td>-0.9918</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>(4,6)</td>
<td>-0.6491</td>
<td>0.9805</td>
<td>-5.1720</td>
<td>4.8069</td>
<td>-4.2355</td>
<td>0.5120</td>
</tr>
<tr>
<td></td>
<td>(8,6)</td>
<td>-0.2548</td>
<td>0.4420</td>
<td>-3.4452</td>
<td>5.0462</td>
<td>-1.4708</td>
<td>0.1293</td>
</tr>
</tbody>
</table>

Furthermore, Figure 5 shows that the discrete calendar-spread approach also exhibits a similar pattern, that is, hedging portfolio values are lower than the target values at the barrier $L$. As an additional information, Table 5 in the appendix displays the static hedge weights at time 0 used to plot Figures 4 and 5.

Figure 4: (Left) Value of the target $\Pi_{0.5}^0(S_{0.5}; 1)$ at time 0.5 year and its discrete static hedge $\Pi_{0.5}^0(S_{0.5}; 0.5)$ with 8 strikes. The embedded barrier option in the latter expired with no knock-in. (Right) Their mismatch values: hedging portfolio value minus option value.

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6It is a provable fact that the discrete strike-spread portfolio yields a sub-replicating portfolio when the target payoff is convex.
Figure 5: (Left) Values of the embedded barrier option $P^{d-i'}$ and its discrete static hedge $P^{d-i'^*}$ with 6 maturities at barrier $L$. (Right) their mismatch values: hedging portfolio value minus option value.

7 Conclusion

This paper developed an algorithm to obtain a static hedging portfolio for a structured product with autocallable, reverse convertible and barrier features. This algorithm can also be applied to diverse structured products under a general Markovian diffusion setting. Numerical experiments showed that static hedging portfolios accurately replicate the target products. Hedging performance was checked under different measures such as mean, standard deviation, min/max, and quantiles.

The core idea of the proposed algorithm is to recursively utilize strike-spread approach and calendar-spread approach. Confining each iteration to the time interval $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$, we decomposed the structured product into European claims and path-dependent options, which are statically hedged via the two options replication approaches. In doing so, we extended the applicability of the strike-spread approach to general European claims with discontinuous payoffs and we statically replicated embedded exotic barrier options using integral equations. Also, to enhance computational and practical feasibilities, we proposed a discrete version of static hedges and provided some convergence analysis.

Some interesting and important questions are left unanswered. There is still a lack of understanding in hedging structured products on multiple underlying assets. Static replication based on our proposal requires basic options values which depend on multiple assets. Then, one can investigate such options values via copula construction or one could approach from sub-/super-
replication viewpoint. Secondly, an empirical study on hedging performances for actual historical data of considered structured products is of considerable interest.

Acknowledgement

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References


Appendix

Laplace transform for \( P_{d-1}^{t_n}(x; t_{n-1}, L, \varphi_n^* - \varphi_n^0), t \in [t_{n-1}, \min\{t_n, \tau\}], \tau = \inf\{u > t_{n-1} : S_u = L\} \).

The left hand side of (2) is equal to \( P_{d-1}^{t_n-t}(x; t_n-t, L, \varphi_n^* - \varphi_n^0) \) and we have

\[
\hat{P}_{d-1}^{t_n}(\lambda, x; L, \varphi_n^* - \varphi_n^0) = \int_0^\infty e^{-\lambda t} P_{d-1}^{t_n-t}(x; t, L, \varphi_n^* - \varphi_n^0) dt
\]

\[
= \int_0^\infty e^{-\lambda t} \int_0^t w(u) P_{d-1}^{t_n-t}(x; t, L, \varphi_n^* - \varphi_n^0) du dt
\]

\[
= \int_0^\infty e^{-\lambda u} w(u) \int_u^\infty e^{-\lambda(t-u)} P_{d-1}^{t_n-t}(x; t-u, L) du dt.
\]
and we have \( \hat{w}(\lambda) = \hat{P}^{d-i}(\lambda, L; L, \varphi_n^0 - \varphi_n^0)/\hat{P}^{eur}(\lambda, L; L) \). In particular, \( \hat{P}^{d-i}(\lambda, L; L, \varphi_n^0 - \varphi_n^0) \) is computed as

\[
\hat{P}^{d-i}(\lambda, L; L, \varphi_n^0 - \varphi_n^0) = \frac{\hat{\nu}(\lambda, L; \varphi_n^0) - \hat{\nu}(\lambda, L; \varphi_n^0)}{(F + r_n)\hat{C}_{bin}(\lambda, L; U_n) + U_n\hat{P}^{bin}(\lambda, L; U_n) - \hat{P}^{eur}(\lambda, L; U_n)} - (F + r_n)\hat{C}_{bin}(\lambda, L; L) - L\hat{P}^{bin}(\lambda, L; L) + \hat{P}^{eur}(\lambda, L; L).
\]

**Proof of Theorem 3:** Firstly, in (5) we easily see that

\[
w_m\Delta_m = -\frac{\varphi(U-) - \varphi_{m-1}}{\Delta_m} = -\varphi'(U-) + O(|\Delta_m|).
\]

This handles the second term in (3).

Next, one can readily check that, for \( i = 1, \ldots, m - 1 \),

\[
w_i = \frac{1}{\Delta_i^2} \left[ \frac{\Delta_i}{\Delta_{i+1}} \varphi_{i+1} - \left(1 + \frac{\Delta_i}{\Delta_{i+1}}\right) \varphi_i + \varphi_{i-1} \right].
\]

As often done in analyzing a finite difference scheme on a nonuniform grid, we assume a smooth strictly increasing mapping \( \iota : [0, U] \to [0, U] \) such that \( \iota(ih) = k_i \) for \( i = 0, \ldots, m \) where \( h = U/m \).

Define \( f(x) = \varphi(\iota(x)) \) and consider the finite difference approximations for \( f'(ih) \) and \( f''(ih) \). Using forward difference for \( f' \), we get

\[
\varphi''(i)h^2 = \left(1 - \frac{\iota''}{\iota'} h\right) \varphi_{i+1} - \left(2 - \frac{\iota''}{\iota'} h\right) \varphi_i + \varphi_{i-1} + O(h^4) + \frac{\iota''}{\iota'} O(h^2).
\]

Here the dependence of \( \iota', \iota'' \) on \( ih \) is suppressed for compact presentation. Assume that \( \iota' = \frac{\Delta_i}{h} \) and \( \iota'' = \frac{\Delta_i}{h^2} \cdot \frac{\Delta_i-\Delta_i+\Delta_{i+1}}{\Delta_{i+1}} \). It is indeed possible to find such a function with two given derivatives. This gives us

\[
\varphi''_i = w_i + \frac{1}{\Delta_i^2} O(h^4) + \frac{1}{\Delta_i^2} \left(1 - \frac{\Delta_i}{\Delta_{i+1}}\right) O(h^2) = w_i + O(h).
\]

Lastly, we observe that the integral part of (3) is approximated by

\[
\int_0^U \varphi''(k)P^{eur}_t(x; T, k)dk = \int_0^{k_{m-1}} \varphi''(k)P^{eur}_t(x; T, k)dk + O(\Delta_m)
\]

\[
= \sum_{i=1}^{m-1} \varphi''_iP^{eur}_t(x; T, k_i)\Delta_i + \varepsilon + O(\Delta_m)
\]

\[
= \sum_{i=1}^{m-1} w_iP^{eur}_t(x; T, k_i)\Delta_i + O(h) + \varepsilon + O(\Delta_m)
\]

where \( \varepsilon \leq \max_{k \in [k_{i-1}, k_i]} |\varphi''(k)P^{eur}_t(x; T, k) - \varphi''_iP^{eur}_t(x; T, k_i)| \) converges to zero as the mesh size decreases. This convergence is uniform in \([0, T] \times \mathbb{R}_+\) thanks to the continuity and boundedness of \( \varphi'' \) and \( P^{eur} \). \( \blacksquare \)
Proof of Theorem 4: For notational convenience, we denote $u_i - w(T - \tau_i) = e_i$ for $i = 1, \ldots, l$. We also write $P^{\text{bin}}_t(T)$ for $P^{\text{bin}}_t(L; T, L)$. Then, observe that, for $i = 0, 1, \ldots, l - 1,$

$$\sum_{j=i+1}^{l} e_j P^{\text{bin}}_{\tau_i}(\tau_j) \Delta_j = v_{\tau_i} - \sum_{j=i+1}^{l} w(T - \tau_j) P^{\text{bin}}_{\tau_i}(\tau_j) \Delta_j$$

$$= \int_{\tau_i}^{T} w(T - u) P^{\text{bin}}_{\tau_i}(u) du - \sum_{j=i+1}^{l} w(T - \tau_j) P^{\text{bin}}_{\tau_i}(\tau_j) \Delta_j.$$

Let us denote the right hand side of the above equation by $\delta_i$. If we consider this equation for index $i - 1$, the difference between two equations lead us to

$$e_i P^{\text{bin}}_{\tau_{i-1}}(\tau_i) \Delta_i + \sum_{j=i+1}^{l} e_j \left[ P^{\text{bin}}_{\tau_{i-1}}(\tau_j) - P^{\text{bin}}_{\tau_i}(\tau_j) \right] \Delta_j = \delta_{i-1} - \delta_i. \quad (9)$$

On the other hand, it can be checked that

$$\frac{\delta_{i-1} - \delta_i}{\Delta_i} = \int_{\tau_{i-1}}^{\tau_i} w(T - u) \frac{P^{\text{bin}}_{\tau_{i-1}}(u) - P^{\text{bin}}_{\tau_i}(u)}{\Delta_i} du - \sum_{j=i+1}^{l} w(T - \tau_j) \frac{P^{\text{bin}}_{\tau_{i-1}}(\tau_j) - P^{\text{bin}}_{\tau_i}(\tau_j)}{\Delta_i} \Delta_j$$

$$+ \int_{\tau_{i-1}}^{\tau_i} w(T - u) \frac{P^{\text{bin}}_{\tau_{i-1}}(u)}{\Delta_i} du - w(T - \tau_{i-1}) P^{\text{bin}}_{\tau_{i-1}}(\tau_i).$$

The expression in the second line converges to zero as the mesh size decreases because $w$ and $P^{\text{bin}}$ are continuous. The first line can be approximated by

$$\int_{\tau_{i-1}}^{\tau_i} w(T - u) \Theta^{\text{bin}}_{\tau_{i-1}}(u) du - \sum_{j=i+1}^{l} w(T - \tau_j) \Theta^{\text{bin}}_{\tau_{i-1}}(\tau_j) \Delta_j,$$

which converges to zero due to the weak singularity of $\Theta$ in Assumption 3.

Consequently, (9) implies

$$|e_i| P^{\text{bin}}_{\tau_{i-1}}(\tau_i) \leq \sum_{j=i+1}^{l} |e_j| |\Theta^{\text{bin}}_{\tau_{i-1}}(\tau_j)| \Delta_j + \varepsilon(\Delta)$$

for some positive function $\varepsilon(\Delta)$ such that $\lim_{\Delta \to 0} \varepsilon(\Delta) = 0$. On the other hand, from the weak singularity of theta, we obtain

$$|\Theta^{\text{bin}}_{\tau_{i-1}}(\tau_j)| \Delta_j = |f(\tau_{i-1}, \tau_i)| \Delta_i^{1-\alpha} \frac{\Delta_j}{\Delta_i}$$

$$= |f(\tau_{i-1}, \tau_i)| \Delta_i^{1-\alpha} (1 + O(1^{-1}))$$

$$= O(\Delta_i^{1-\alpha}).$$

Since $P^{\text{bin}}_{\tau_{i-1}}(\tau_i)$ is close to some non-zero constant for small $\Delta_i$, we can conclude that

$$|e_i| \leq \varepsilon'(\Delta) \sum_{j=i+1}^{l} |e_j| + \varepsilon(\Delta)$$

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Table 5: Static hedging portfolio for step down ELS at time 0: $\Pi_0^* = v_0^* \pm P_0^{d-i^*}$.

<table>
<thead>
<tr>
<th>product $v_0^*(x; t_1, \varphi^{0\ell}_1)$</th>
<th>type $C^{bin}$</th>
<th>strike 95</th>
<th>maturity 0.5</th>
<th>weight 102</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P^{bin}$</td>
<td>strike 95</td>
<td>maturity 0.5</td>
<td>weight 94.647</td>
</tr>
<tr>
<td></td>
<td>$P^{d-i^*}(x; t_1, L, \varphi^{0\ell}_1 - \varphi^{0\ell}_1)$</td>
<td>strike 55</td>
<td>maturity $\frac{1}{12}$</td>
<td>weight $-0.0003$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>strike 55</td>
<td>maturity $\frac{2}{12}$</td>
<td>weight $-0.0007$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>strike 55</td>
<td>maturity $\frac{3}{12}$</td>
<td>weight $-0.0017$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>strike 55</td>
<td>maturity $\frac{4}{12}$</td>
<td>weight $-0.0041$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>strike 55</td>
<td>maturity $\frac{5}{12}$</td>
<td>weight $-0.0106$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>strike 55</td>
<td>maturity $\frac{6}{12}$</td>
<td>weight $-0.2187$</td>
</tr>
</tbody>
</table>

for some positive function $\varepsilon'(\Delta)$ which decreases to zero for decreasing mesh size. Gronwall’s inequality then implies that there exists a constant $M$ with

$$|e_t| \leq M \left( \varepsilon(\Delta) + \varepsilon'(\Delta)|e_t| \right).$$

With this established convergence of $u_t$ to $w(T - \cdot)$, it is now easy to see

$$\int_t^T w(T - u)P^{bin}_t(x; u, L)du \approx \sum_{j=i+1}^l w(T - \tau_j)P^{bin}_{\tau_j}(x; \tau_j, L)\Delta_j$$

$$\approx \sum_{j=i+1}^l u_j P^{bin}_{\tau_j}(x; \tau_j, L)\Delta_j$$

where $i$ is chosen so that $\tau_{i-1} \leq t \leq \tau_i$. The proof is complete. □

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